Genuinely multi-dimensional equilibria preservation on curved domains with a ghost point method for acoustic equations

Davide Torlo, Mirco Ciallella, Mario Ricchiuto

Dipartimento di Matematica "Guido Castelnuovo", Università di Roma La Sapienza, Italy
davidetorlo.it

13th International Conference on Computational Fluid Dynamics (ICCFD13)
Milan, Italy – 8th July 2026

SAPIENZA
UNIVERSITÀ DI ROMA

Table of contents

Moving equilibria and perturbations (shallow water example)

- Lake at rest perturbation
- Moving stationary wave
- Vortex type stationary solutions

Hyperbolic balance laws

Balance laws

$$\partial_t q(t,x,y) + \partial_x f(q(t,x,y)) + \partial_y g(q(t,x,y)) = S(q(t,x,y),t,x,y), \tag{1}$$

Moving Equilibria

$$\partial_x f(q(t,x,y)) + \partial_y g(q(t,x,y)) = S(q(t,x,y),t,x,y), \tag{2}$$

Acoustic equations

For this system $q=(p,u,v)$, where p is the pressure and u,v are the velocity components, while in the simplest setting the fluxes and source term can be written as

$$f(q) = \begin{pmatrix} u \\ p \\ 0 \end{pmatrix}, \qquad g(q) = \begin{pmatrix} v \\ 0 \\ p \end{pmatrix}, \qquad S(q,t,x,y) = 0. \tag{3}$$

$$\text{Equilibrium: } \partial_x u + \partial_y v = 0 \qquad p \equiv p_0. \tag{4}$$

Multidimensional formulation

Integrating fluxes and sources

$$\partial_t q(t,x,y) + \partial_x f(q(t,x,y)) + \partial_y g(q(t,x,y)) = S(q(t,x,y),t,x,y), \tag{5}$$

$$\partial_t q(t,x,y) + \partial_x \partial_y \mathscr{F}(t,x,y) = 0, \tag{6}$$

where $\mathscr{F}$ is the **global flux**[1] and it is defined as

$$\mathscr{F}(t,x,y) = \int_{y_b}^{y} f(q(t,x,\eta))\mathrm{d}\eta + \int_{x_\ell}^{x} g(q(t,\xi,y))\mathrm{d}\xi - \int_{x_\ell}^{x}\int_{y_b}^{y} S(q(t,\xi,\eta),t,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta. \tag{7}$$

New Equilibria Formulation (Simpler form)

$$\partial_x \partial_y \mathscr{F} = 0 \iff \mathscr{F}(x,y) = \sigma(x) + \gamma(y) \tag{8}$$

[1] Inspired by 1D framework: Gascón (2002), Chertock (2018)

Finite Volume Corner-Flux Formulation

Equation

$\partial_t q(t,x,y) + \partial_x \partial_y \mathscr{F}(t,x,y) = 0,$

Finite Volume Formulation

$$\frac{q_{i,j}^{n+1} - q_{i,j}^n}{\Delta t} + \frac{\mathscr{F}(t^n, x_{i+\frac12}, y_{j+\frac12}) - \mathscr{F}(t^n, x_{i-\frac12}, y_{j+\frac12}) - \mathscr{F}(t^n, x_{i+\frac12}, y_{j-\frac12}) + \mathscr{F}(t^n, x_{i-\frac12}, y_{j-\frac12})}{\Delta x \Delta y} = 0. \quad (9)$$

$$\widehat{\mathscr{F}}^{(i,j)}_{i+\frac12, j+\frac12} \approx \pm \mathscr{F}(t^n, x_{i+\frac12}, y_{j+\frac12}) \quad (10)$$

Corner flux definition[2]

$$\frac{q_{i,j}^{n+1} - q_{i,j}^n}{\Delta t} + \frac{\widehat{\mathscr{F}}^{(i,j)}_{i+\frac12, j+\frac12} + \widehat{\mathscr{F}}^{(i+1,j+1)}_{i+\frac12, j+\frac12} + \widehat{\mathscr{F}}^{(i+1,j)}_{i+\frac12, j+\frac12} + \widehat{\mathscr{F}}^{(i,j+1)}_{i+\frac12, j+\frac12}}{\Delta x \Delta y} = 0. \quad (11)$$

[2] Barsukow, Ciallella, Riucchiuto, Torlo. 2026.

Corner Flux Sketch (Dual Cell SUPG formulation)

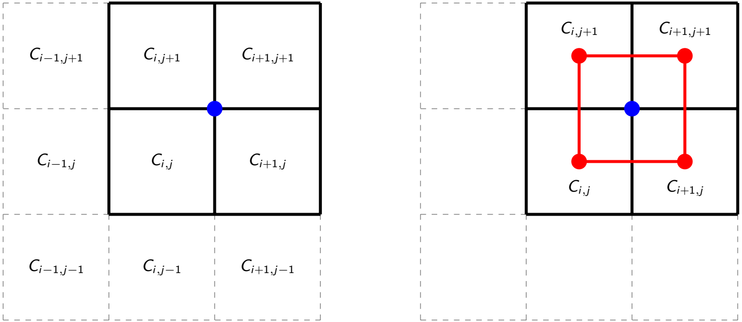


Figure: Cell labeling for the 2D grid: main grid (left), dual grid (right).

SUPG Corner Flux

- SUPG Corner Fluxes

$$\widehat{\mathscr{F}}^{(i,j+1)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i,j+1)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i,j+1)}_{i+\frac12,j+\frac12}, \qquad \widehat{\mathscr{F}}^{(i+1,j+1)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i+1,j+1)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i+1,j+1)}_{i+\frac12,j+\frac12},$$

$$\widehat{\mathscr{F}}^{(i,j)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i,j)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i,j)}_{i+\frac12,j+\frac12}, \qquad \widehat{\mathscr{F}}^{(i+1,j)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i+1,j)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i+1,j)}_{i+\frac12,j+\frac12},$$

- Central Part

$$\overline{\mathscr{F}}_{i+\frac12,j+\frac12} = \frac{1}{4}(\mathscr{F}_{i,j} + \mathscr{F}_{i+1,j} + \mathscr{F}_{i+1,j+1} + \mathscr{F}_{i,j+1})$$

- Diffusion Part

$$\mathcal{D}^{(i,j+1)}_{i+\frac12,j+\frac12} = \frac{\alpha\Delta}{4}\left(-\frac{J^x}{\Delta x} + \frac{J^y}{\Delta y}\right)\Phi_{i+\frac12,j+\frac12}, \qquad \mathcal{D}^{(i+1,j+1)}_{i+\frac12,j+\frac12} = \frac{\alpha\Delta}{4}\left(+\frac{J^x}{\Delta x} + \frac{J^y}{\Delta y}\right)\Phi_{i+\frac12,j+\frac12},$$

$$\mathcal{D}^{(i,j)}_{i+\frac12,j+\frac12} = \frac{\alpha\Delta}{4}\left(-\frac{J^x}{\Delta x} - \frac{J^y}{\Delta y}\right)\Phi_{i+\frac12,j+\frac12}, \qquad \mathcal{D}^{(i+1,j)}_{i+\frac12,j+\frac12} = \frac{\alpha\Delta}{4}\left(+\frac{J^x}{\Delta x} - \frac{J^y}{\Delta y}\right)\Phi_{i+\frac12,j+\frac12},$$

$$\Phi_{i+\frac12,j+\frac12} := \Phi(\widetilde{\mathscr{F}}_{i+\frac12,j+\frac12}) := \mathscr{F}_{i+1,j+1} - \mathscr{F}_{i,j+1} - \mathscr{F}_{i+1,j} + \mathscr{F}_{i,j} = \int_{\widetilde{C}_{i+\frac12,j+\frac12}} \partial_{xy}\widetilde{\mathscr{F}}_{i+\frac12,j+\frac12}\mathrm{d}x\mathrm{d}y\,.$$

- When $\mathscr{F}_{i,j} = \sigma_i + \gamma_j$ **everything cancels out!**

Comparison with Rusanov Corner Flux

- SUPG Corner Fluxes

$$\widehat{\mathscr{F}}^{(i,j+1)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i,j+1)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i,j+1)}_{i+\frac12,j+\frac12}, \qquad \widehat{\mathscr{F}}^{(i+1,j+1)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i+1,j+1)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i+1,j+1)}_{i+\frac12,j+\frac12},$$
$$\widehat{\mathscr{F}}^{(i,j)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i,j)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i,j)}_{i+\frac12,j+\frac12}, \qquad \widehat{\mathscr{F}}^{(i+1,j)}_{i+\frac12,j+\frac12} = \overline{\mathscr{F}}_{i+\frac12,j+\frac12} n^{(i+1,j)}_{i+\frac12,j+\frac12} + \mathcal{D}^{(i+1,j)}_{i+\frac12,j+\frac12},$$

- Central Part

$$\overline{\mathscr{F}}_{i+\frac12,j+\frac12} = \frac{1}{4}\left(\mathscr{F}_{i,j} + \mathscr{F}_{i+1,j} + \mathscr{F}_{i+1,j+1} + \mathscr{F}_{i,j+1}\right)$$

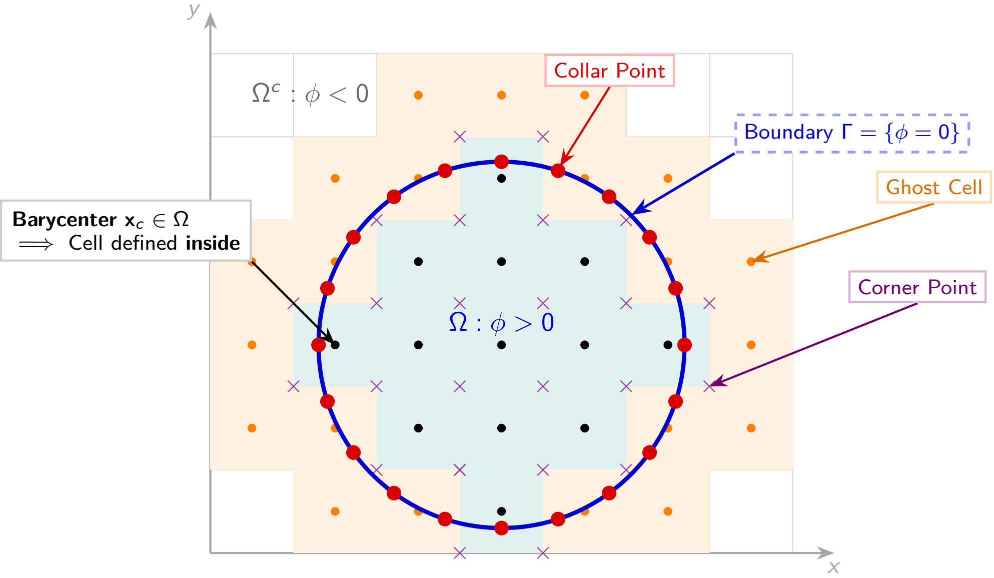
- Diffusion Part

$$\mathcal{D}^{(i,j+1)}_{i+\frac12,j+\frac12} = \alpha\Delta(q_{i,j+1} - \bar{q}_{i+\frac12,j+\frac12}), \qquad \mathcal{D}^{(i+1,j+1)}_{i+\frac12,j+\frac12} = \alpha\Delta(q_{i+1,j+1} - \bar{q}_{i+\frac12,j+\frac12}),$$
$$\mathcal{D}^{(i,j)}_{i+\frac12,j+\frac12} = \alpha\Delta(q_{i,j} - \bar{q}_{i+\frac12,j+\frac12}), \qquad \mathcal{D}^{(i+1,j)}_{i+\frac12,j+\frac12} = \alpha\Delta(q_{i+1,j} - \bar{q}_{i+\frac12,j+\frac12}),$$
$$\bar{q}_{i+\frac12,j+\frac12} = \frac{1}{4}\left(q_{i,j} + q_{i+1,j} + q_{i+1,j+1} + q_{i,j+1}\right)$$

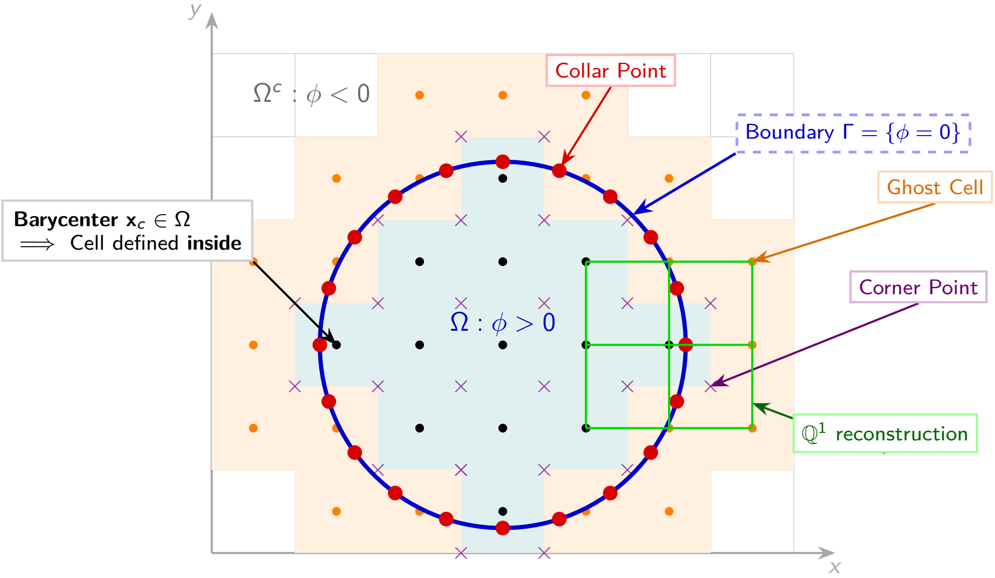
- When $\mathscr{F}_{i,j} = \sigma_i + \gamma_j$ **the diffusion part might still contribute!**

Table of contents

Geometric Classifications: How to set ghosts cells?



Geometric Classifications: How to set ghosts cells?



Extrapolation from the Interior and Collar Points

Least Square Problem (Dirichlet BC)

$$\min_{p(x,y)\in\mathbb{Q}^1} \sum_{\ell=-1}^{1}\sum_{r=-1}^{1} \left(p(x_{\tilde{i}+\ell}, y_{\tilde{j}+r}) - q(x_{\tilde{i}+\ell}, y_{\tilde{j}+r})\right) \chi(x_{\tilde{i}+\ell}, y_{\tilde{j}+r})_{\tilde{\Omega}} \qquad \text{subject to}$$
$$p(x_k^c, y_k^c) = u(x_k^c, y_k^c), \qquad (x_k^c, y_k^c) \in \Gamma \text{ collar points for } k = 1, \dots, N^c_{i+\frac{1}{2}, j+\frac{1}{2}}, \tag{12}$$

where (x_k^c, y_k^c) are the closest $N^c_{i+\frac{1}{2}, j+\frac{1}{2}} \in \mathbb{N}$ collar points to the corner point $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ and $\chi(x_{\tilde{i}+\ell}, y_{\tilde{j}+r})_{\tilde{\Omega}}$ is the indicator function that is 1 if the cell with barycenter $(x_{\tilde{i}+\ell}, y_{\tilde{j}+r})$ is in the computational domain and 0 otherwise.

Computational Aspects

- Lagrange Multiplier Method
- Least Square Matrices are small
- Cheap
- **Second Order Accurate**
- Can be computed at the beginning of the simulation (and inverted)
- Number of collar points chosen to stabilize and make solvable the system

Extension to Neumann BC and Vector BC

Neumann BC

$$\min_{p(x,y)\in\mathbb{Q}^1} \sum_{\ell=-1}^{1}\sum_{r=-1}^{1} \left(p(x_{\tilde{i}+\ell}, y_{\tilde{j}+r}) - q(x_{\tilde{i}+\ell}, y_{\tilde{j}+r})\right) \chi(x_{\tilde{i}+\ell}, y_{\tilde{j}+r})_{\tilde{\Omega}} \qquad \text{subject to}$$
$$\nabla p(x_k^c, y_k^c)\cdot \mathbf{n}(x_k^c, y_k^c) = g(x_k^c, y_k^c), \qquad (x_k^c, y_k^c)\in\Gamma \text{ collar points for } k=1,\dots,N^c_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{13}$$

where g is the function that defines the Neumann boundary condition, i.e., $\nabla u\cdot\mathbf{n} = g$ on Γ.

- **First Order Accurate**

Wall BC Collar Points Constraints

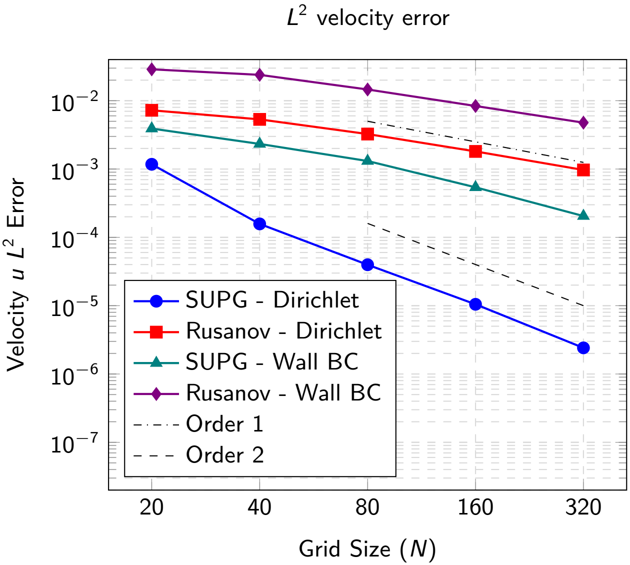
- $\mathbf{v}\cdot\mathbf{n} = 0$
- $\partial_{\mathbf{n}} p = 0$
- $\partial_{\mathbf{n}}(\mathbf{v}\cdot\tau) = 0$
- Larger system, similar blocks
- **First Order Accurate**

Table of contents

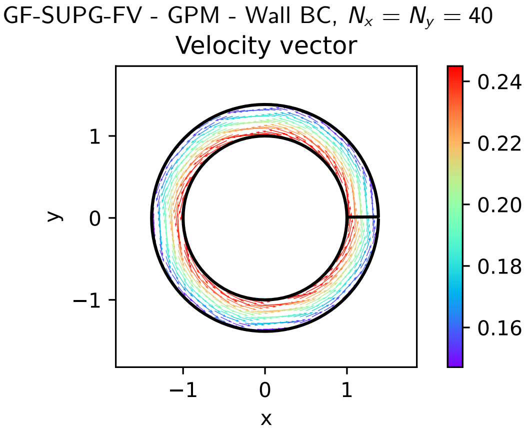
Acoustic vortex in ring

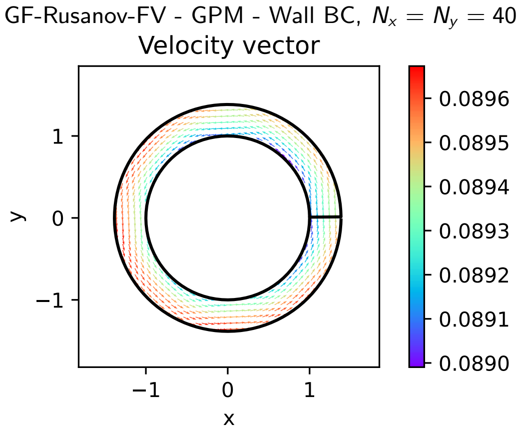
Exact Solution

$$\Omega = \{1 \leq \sqrt{x^2+y^2} \leq 1.384\},$$
$$p(x,y) = 1,$$
$$u(x,y) = yf(\rho(x,y)),$$
$$v(x,y) = -xf(\rho(x,y)),$$
$$r_0 = 2.5,$$
$$\rho(x,y) = \frac{\sqrt{x^2+y^2}}{r_0},$$
$$f(\rho) = \gamma(1+\cos(\pi\rho))^2,$$
$$\gamma = \frac{12\pi\sqrt{0.981}}{r_0\sqrt{315\pi^2-2048}}.$$

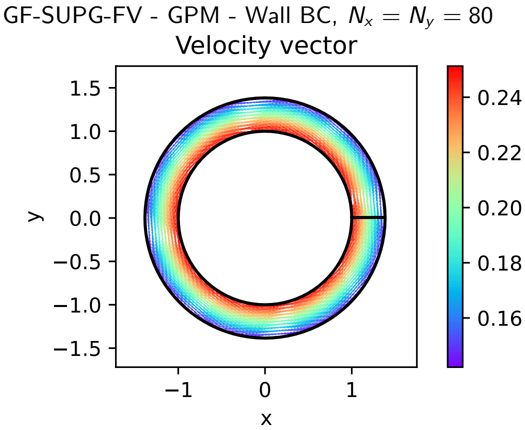


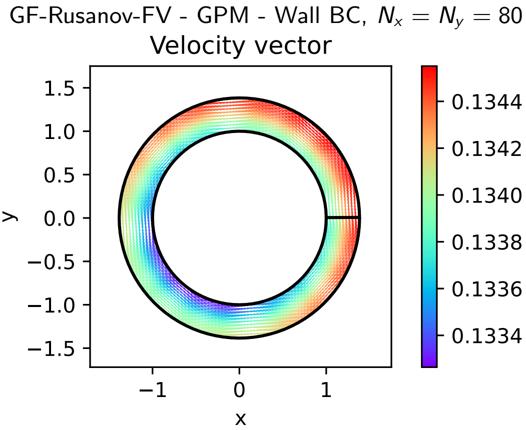
Acoustic vortex in ring





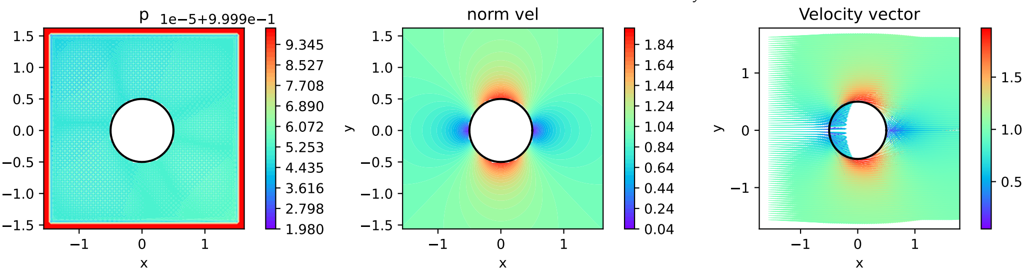
Acoustic vortex in ring



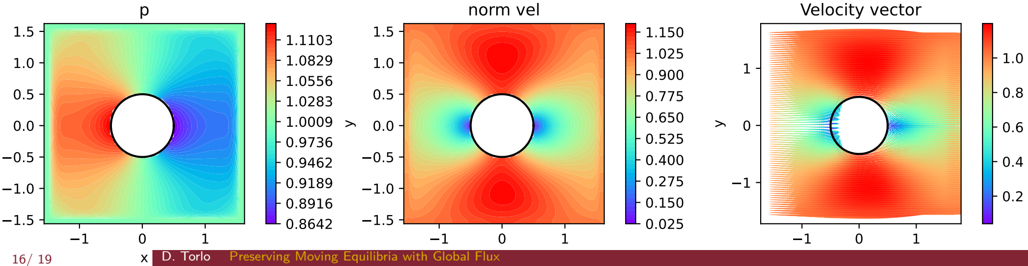


Potential Flow Around Cylinder

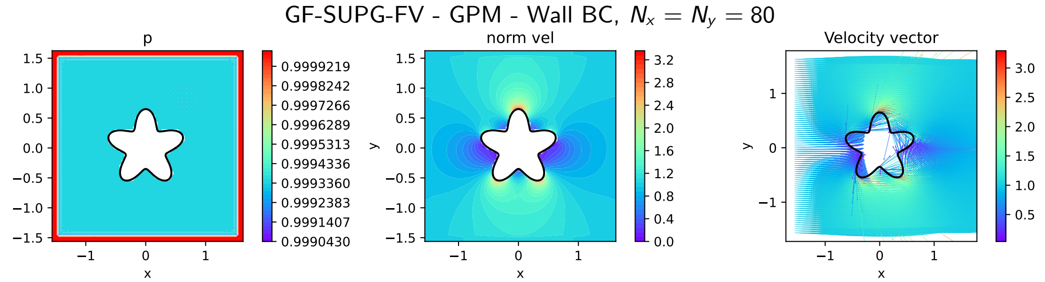
GF-SUPG-FV - GPM - Wall BC, $N_x = N_y = 80$



GF-Rusanov-FV - GPM - Wall BC, $N_x = N_y = 80$



Potential Flow Around Flower



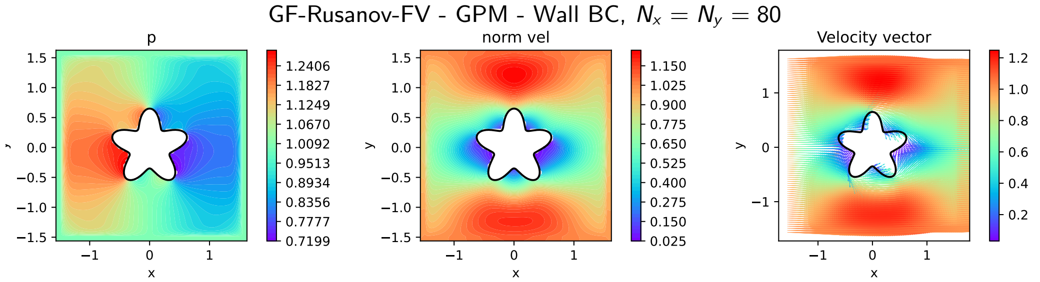


Table of contents

Conclusions

Summary of Work

- Moving Equilibria in Non Cartesian Domains
- Multidimensional Global Flux (Corner Flux)
- Ghost Point Method

Extensions

- Nonlinear problems
- Higher Order Approximations
- 3D Formulations

Conclusions

Summary of Work

- Moving Equilibria in Non Cartesian Domains
- Multidimensional Global Flux (Corner Flux)
- Ghost Point Method

Extensions

- Nonlinear problems
- Higher Order Approximations
- 3D Formulations

THANKS!!