How to Preserve Moving Equilibria: Global Flux Quadrature

Davide Torlo, Wasilij Barsukow, Mirco Ciallella, Mario Ricchiuto, Moussa Ziggaf

Dipartimento di Matematica "Guido Castelnuovo", Università di Roma La Sapienza, Italy
davidetorlo.it

Roma - 15th December 2025

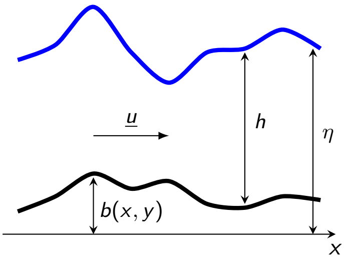
Water equilibria and perturbations

- Lake at rest perturbation
- Moving stationary wave
- Vortex type stationary solutions

Equilibria for shallow water equations

Shallow Water Equations

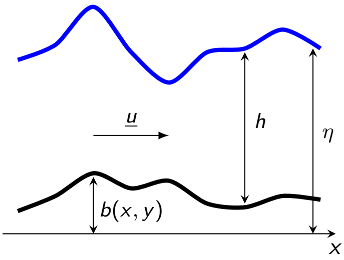
$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y\left(huv\right) = -gh\partial_x b \\ \partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = -gh\partial_y b \end{cases}$$



Equilibria for shallow water equations

Shallow Water Equations

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y(huv) = -gh\partial_x b \\ \partial_t(hv) + \partial_x(huv) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = -gh\partial_y b \end{cases}$$

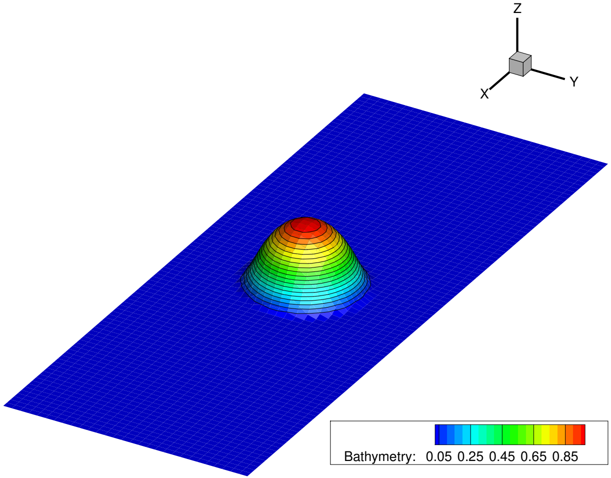


Lake at rest equilibrium

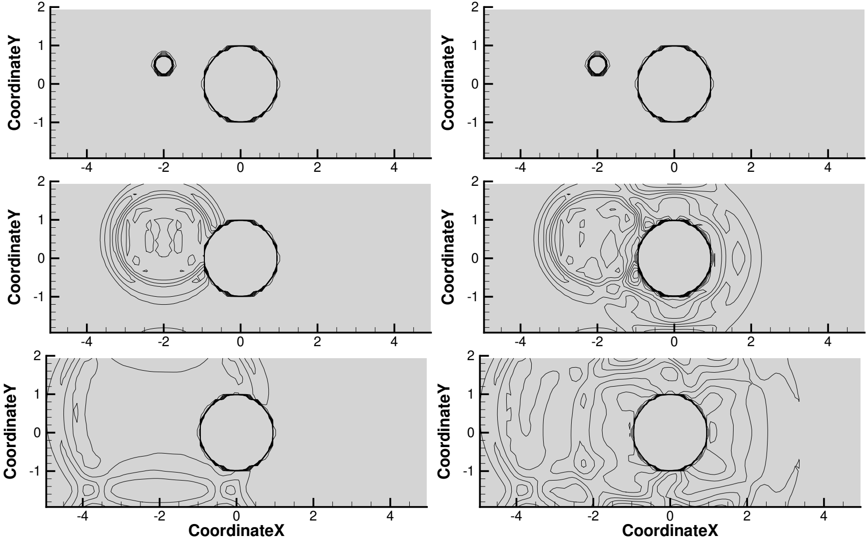
$$h(x,y) + b(x,y) \equiv \eta_0 \qquad u(x,y) = v(x,y) \equiv 0$$

$$\partial_x\left(\frac{g}{2}h^2\right) + gh\partial_x b = gh\partial_x h + gh\partial_x b = gh\partial_x \eta_0 = 0.$$

Simulation example lake at rest with perturbation



Simulation example lake at rest with perturbation



Equilibria for shallow water equations

Shallow Water Equations 1D

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = -gh\partial_x b \end{cases}$$

Stationary waves in 1D

$$hu(x) =: q(x) \equiv q_0^x$$

and h such that

$$\partial_x \left(hu^2 + \frac{g}{2} h^2 \right) + gh\partial_x b = 0$$

$$\dots$$

$$\partial_x \left(\frac{q^2}{2gh^2} + h + b \right) = 0$$

$$\frac{q^2}{2gh^2(x)} + h(x) + b(x) = \mathcal{Q}(x_0) \qquad (1)$$

Equilibria for shallow water equations

Shallow Water Equations 1D

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = -gh\partial_x b \end{cases}$$

Cubic equation solutions

- Supercritical state $u > \sqrt{gh}$
- Subcritical state $u < \sqrt{gh}$
- Negative h

Stationary waves in 1D

$$hu(x) =: q(x) \equiv q_0^x$$

and h such that

$$\partial_x \left(hu^2 + \frac{g}{2} h^2 \right) + gh\partial_x b = 0$$

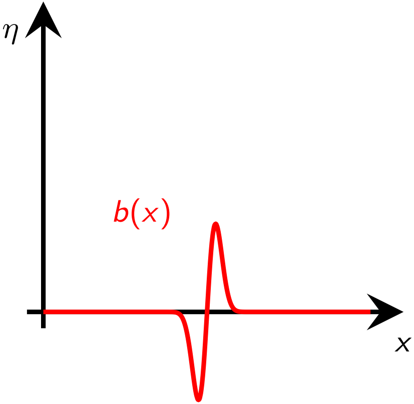
$$\dots$$

$$\partial_x \left(\frac{q^2}{2gh^2} + h + b \right) = 0$$

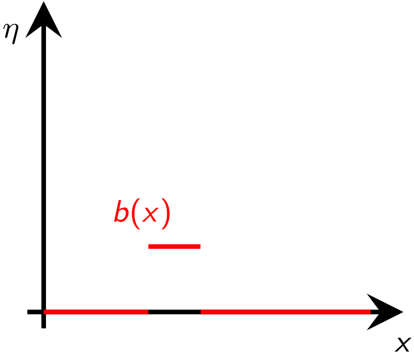
$$\frac{q^2}{2gh^2(x)} + h(x) + b(x) = \mathcal{Q}(x_0) \tag{1}$$

Simulation example moving equilibria non flat bathymetry

Continuous Bathymetry

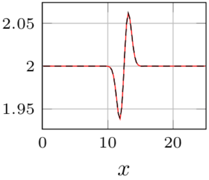


Discontinuous Bathymetry

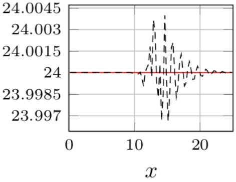


Simulation example moving equilibria non flat bathymetry

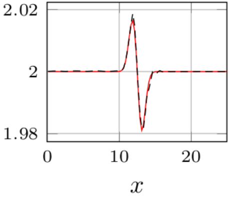
Continuous Bathymetry



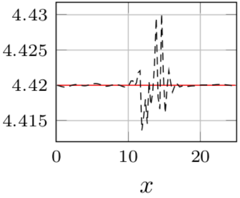
(a) η



(b) q

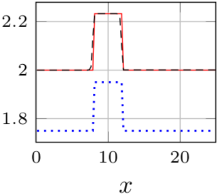


(a) η

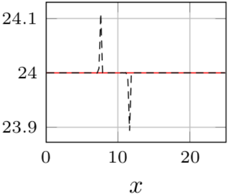


(b) q

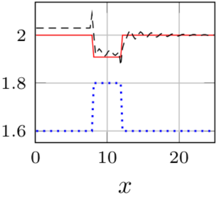
Discontinuous Bathymetry



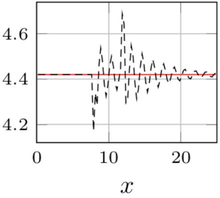
(a) η



(b) q



(a) η



(b) q

2

Equilibria for shallow water equations

Shallow Water Equations (no bathymetry)

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y\left(huv\right) = 0 \\ \partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = 0 \end{cases}$$

Vortices: Div-free solutions

$$\begin{cases} r = (x-x_0)^2 + (y-y_0)^2 \quad \theta = \arctan\left(\frac{y-y_0}{x-x_0}\right) \\ u(r) = -\sin(\theta)u_\theta(r) \qquad v(r) = \cos(\theta)u_\theta(r) \\ h(r): \quad h'(r)gr = u_\theta^2(r) \end{cases}$$

Other equations

- Euler equations (isentropic)
- Linear Acoustic equations

Simulation example of a vortex (for linear acoustics)

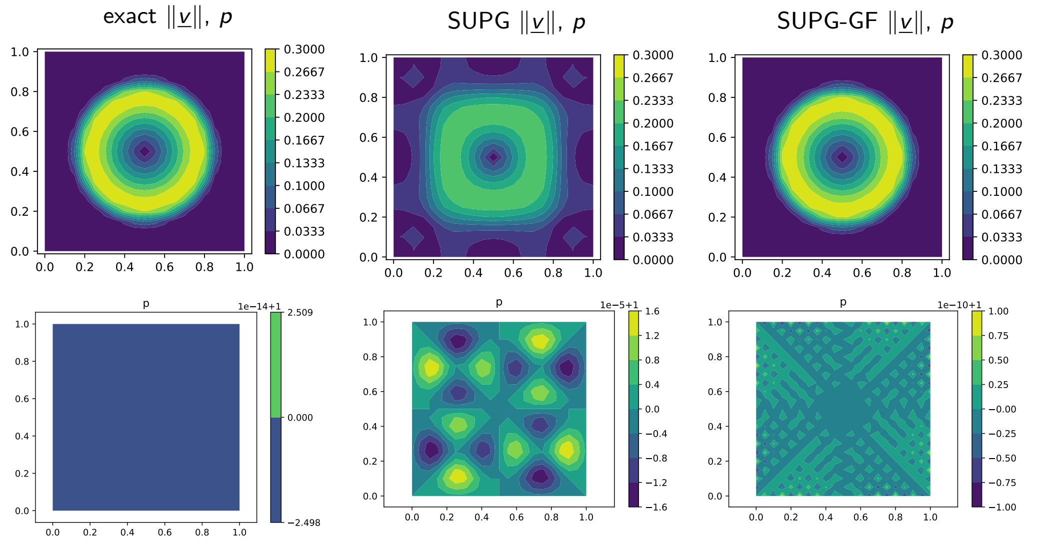


Table of contents

How can we preserve the equilibria?

Exactly!

Impossible: discretization of data b, of the solutions h, u, v

Exactly with respect to discretization

- Possible
- Might involve some analytical equation to be solved
- Requires the knowledge a priori of equilibria type

Exactly Well Balancing

Better than the underlying method

- Possible
- No need of inverting analytical equations
- No need of a priori knowledge of the equilibrium type

Well Balancing

State of the art techniques

Global Flux

- Obtain 1 differential operator for everything
- Put together flux and source
- Integrate the forms
- Gascón 2001[a], Chertock 2022[b], Ciallella 2023[c], Barsukow 2024[d]

[a]Gascón, L., Corberán, J. J. Comput. Phys. 172(1), 261–297 (2001)

[b]Chertock, A., Kurganov, A., Liu, X., Liu, Y., & Wu, T. (2022). Journal of Scientific Computing, 90, 1-21.

[c]Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

[d]Barsukow, W., Ricchiuto, M., & Torlo, D. (2025). Numerical Methods for Partial Differential Equations 41.1 (2025): e23167.

1D source recipe

$$\partial_t V + \partial_x f(V) = S(V,x)$$
$$\partial_t V + \partial_x (f(V) - K(V,x)) = 0$$
$$K(V,x) := \int_{x_0}^{x} S(V(s),s)ds$$

2D divergence recipe

$$\partial_t h + \partial_x f + \partial_y g = 0, \qquad f = hu, \quad g = hv,$$
$$\partial_t h + \partial_{xy}(F+G) = 0$$
$$F(x,y) := \int_{y_0}^{y} f(x,\xi)d\xi, \quad G(x,y) := \int_{x_0}^{x} g(\xi,y)d\xi.$$

State of the art techniques

Global Flux

- Obtain 1 differential operator for everything
- Put together flux and source
- Integrate the forms
- Gascón 2001[a], Chertock 2022[b], Ciallella 2023[c], Barsukow 2024[d]

[a] Gascón, L., Corberán, J. J. Comput. Phys. 172(1), 261–297 (2001)

[b] Chertock, A., Kurganov, A., Liu, X., Liu, Y., & Wu, T. (2022). Journal of Scientific Computing, 90, 1-21.

[c] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

[d] Barsukow, W., Ricchiuto, M., & Torlo, D. (2025). Numerical Methods for Partial Differential Equations 41.1 (2025): e23167.

Properties

- ☹ Well balanced (not exactly)
- ☺ No need for any analytical equilibria
- ☺ No need for analytical relation
- ☺ No further ODE solver
- ☺ No problems with transcritical points
- ☺ Explicit methods
- ☺ Lake at rest
- ☺ Stationary waves
- ☺ 2D vortices
- ☺ Applicable to FV, FEM, DG

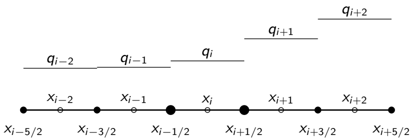
Table of contents

Global Flux in 1D for FV 1st order

$$\partial_t q + \partial_x f(q) = S(q,x) \qquad \Longrightarrow \qquad \partial_t q + \partial_x G(q,x) = 0$$

with $$G(q,x) := f(q) - K(q,x) = f(q) - \int_{x_0}^{x} S(q(s),s)\mathrm{d}s.$$

$$\text{FV: } q_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\mathrm{d}x$$

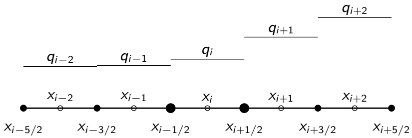


Global Flux in 1D for FV 1st order

$$\partial_t q + \partial_x f(q) = S(q,x) \qquad \Longrightarrow \qquad \partial_t q + \partial_x G(q,x) = 0$$

$$\text{with } G(q,x) := f(q) - K(q,x) = f(q) - \int_{x_0}^{x} S(q(s),s)\mathrm{d}s.$$

$$\text{FV: } q_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\mathrm{d}x$$



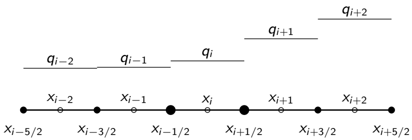
$$\begin{aligned}
f_i :=& \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\mathrm{d}x,\\
K_i \approx& K(x_i, q(x_i)) = \int_{x_0}^{x_i} S(q(s),s)\mathrm{d}s \approx K_{i-1} + \int_{x_{i-1}}^{x_i} S(q(s),s)\mathrm{d}s,\\
K_i :=& K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}\\
G_i :=& f_i - K_i.
\end{aligned}$$

Global Flux in 1D for FV 1st order

$$\partial_t q + \partial_x f(q) = S(q,x) \qquad \Longrightarrow \qquad \partial_t q + \partial_x G(q,x) = 0$$

$$\text{with } G(q,x) := f(q) - K(q,x) = f(q) - \int_{x_0}^{x} S(q(s),s)\mathrm{d}s.$$

$$\text{FV: } q_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\mathrm{d}x$$



$$\begin{aligned}
f_i :=& \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\mathrm{d}x,\\
K_i \approx& K(x_i, q(x_i)) = \int_{x_0}^{x_i} S(q(s),s)\mathrm{d}s \approx K_{i-1} + \int_{x_{i-1}}^{x_i} S(q(s),s)\mathrm{d}s,\\
K_i :=& K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}\\
G_i :=& f_i - K_i.
\end{aligned}$$

Numerical flux depends only on G: upwind, Roe, NO Rusanov

$$\partial_t q_i + \frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} = 0,$$

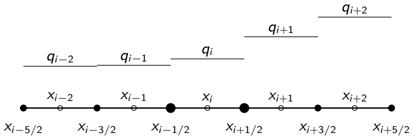
$$\hat{G}_{i+1/2} := \mathrm{sign}(J)^+ G_i + \mathrm{sign}(J)^- G_{i+1},$$

Global Flux in 1D for FV 1st order

$$\partial_t q + \partial_x f(q) = S(q,x) \qquad \Longrightarrow \qquad \partial_t q + \partial_x G(q,x) = 0$$

with $G(q,x) := f(q) - K(q,x) = f(q) - \int_{x_0}^{x} S(q(s),s)\mathrm{d}s.$

FV: $q_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\mathrm{d}x$



$$f_i := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\mathrm{d}x,$$

$$K_i \approx K(x_i, q(x_i)) = \int_{x_0}^{x_i} S(q(s),s)\mathrm{d}s \approx K_{i-1} + \int_{x_{i-1}}^{x_i} S(q(s),s)\mathrm{d}s,$$

$$K_i := K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}$$

$$G_i := f_i - K_i.$$

Numerical flux depends only on G: upwind, Roe, NO Rusanov

$$\partial_t q_i + \frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} = 0,$$

$$\hat{G}_{i+1/2} := \mathrm{sign}(J)^+ G_i + \mathrm{sign}(J)^- G_{i+1},$$

Equilibrium: $\hat{G}_{i+1/2} = \hat{G}_{i-1/2} = \hat{G}_0$ for all i

$$f_i - K_i = G_0$$

Mind: high order, other equilibria (LAR), super convergence

Developing GF 1D FV 1st order

I want you to hate me, let's do the computations in a simple case (upwind)!

Formulae

- $\partial_t q_i = -\frac{\hat{G}_{i+1/2}-\hat{G}_{i-1/2}}{\Delta x}$
- $G_i = f_i - K_i$
- $K_i = K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}$
- $\mathrm{sign}(J) = +1$
- $\hat{G}_{i+1/2} = G_i$

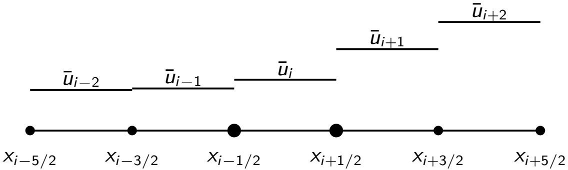
Classical Upwind FV

$$\partial_t q_i = -\frac{f_i - f_{i-1}}{\Delta x} + S_i$$

Expand!

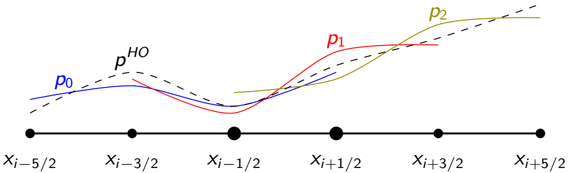
$$\begin{aligned}
\partial_t q_i &= -\frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} \\
&= -\frac{G_i - G_{i-1}}{\Delta x} \\
&= -\frac{f_i - f_{i-1}}{\Delta x} + \frac{K_i - K_{i-1}}{\Delta x} \\
&= -\frac{f_i - f_{i-1}}{\Delta x} + \frac{S_{i-1} + S_i}{2}.
\end{aligned}$$

High order WENO GF [1]



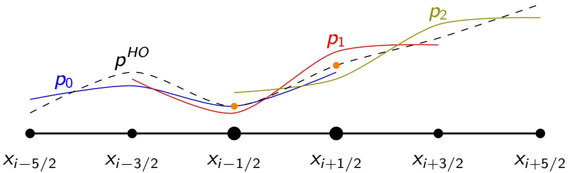
[1] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

High order WENO GF [1]



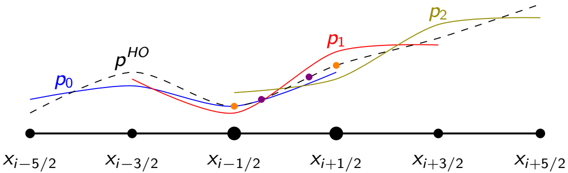
[1] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

High order WENO GF [1]



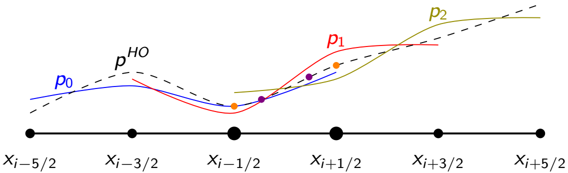
[1] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

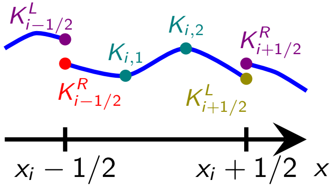
High order WENO GF [1]



[1] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

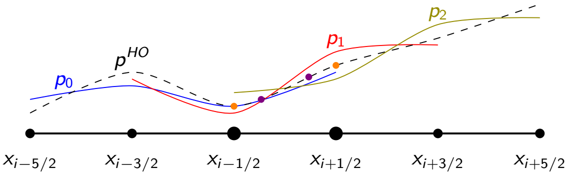
High order WENO GF [1]

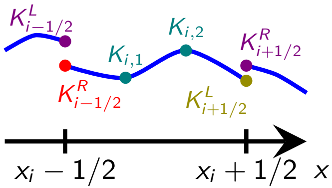




[1] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

High order WENO GF [1]





Global Flux Reconstruction

- Compute recursively K in quadrature points and interfaces (maybe also jump of K)
- Reconstruct in all quadrature points
  - Flux $f_{i,\theta}$
  - Integral of the source $K_{i,\theta}$
  - Global fluxes $G_{i,\theta} := f_{i,\theta} + K_{i,\theta}$
- Compute the cell average of the global flux G
- Well balancing for lake at rest

[1] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

Validation: Subcritical flow and perturbation

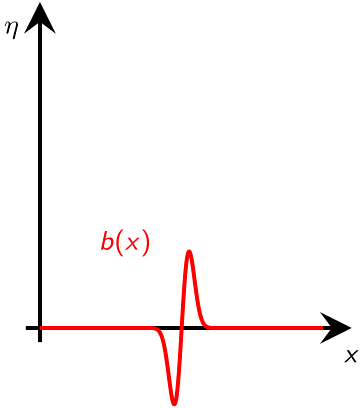
Domain and Bathymetry

$$\Omega = [0, 25],$$
$$b(x) = 0.05 \sin(x - 12.5) \exp\left(1 - (x - 12.5)^2\right),$$
$$g = 9.812.$$

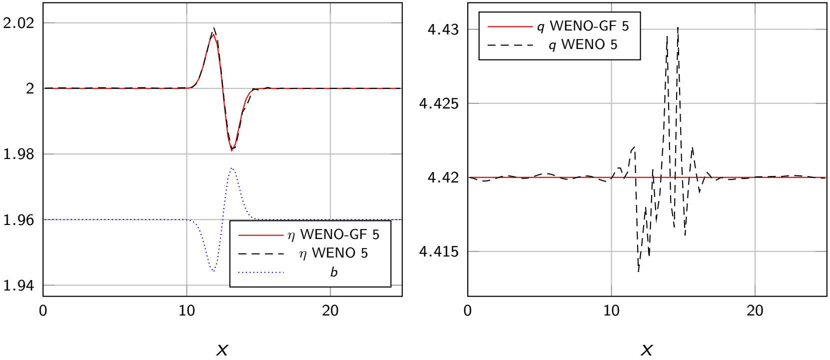
$b(x)$ is chosen $\mathcal{C}^\infty$ and such that it has values smaller than machine precision at the boundaries.

Subcritical flow test

$$\begin{aligned} &\text{IC: } h(x,0) = 2 - b(x), \quad && q(x,0) \equiv 0, \\ &\text{BC: } h(25,t) = 2, && q(0,t) = 4.42, \end{aligned}$$



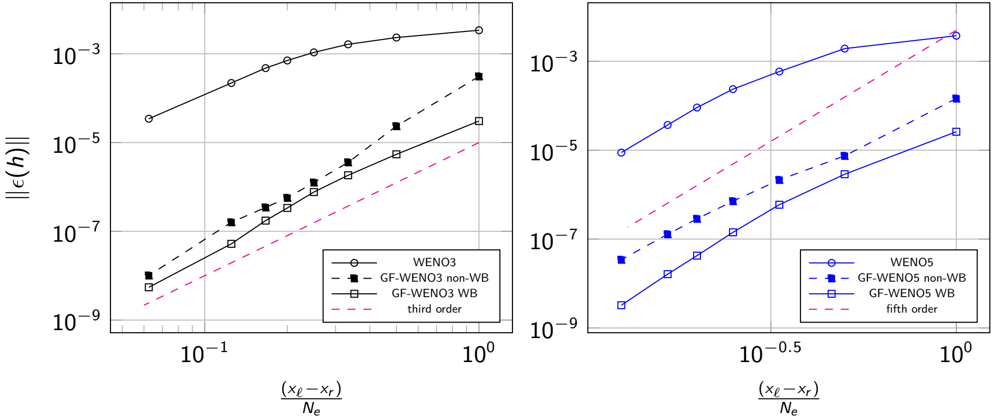
Validation: Subcritical flow and perturbation



(a) η (b) q

Figure: Subcritical flow: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes with $N_e = 100$.

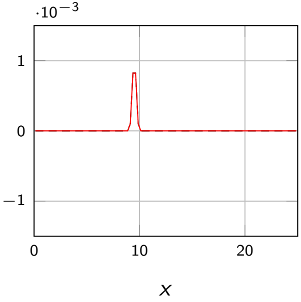
Validation: Subcritical flow and perturbation



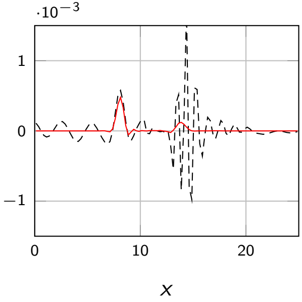
(a) WENO3 (b) WENO5

Figure: Subcritical flow: convergence tests with WENO3 and WENO5.

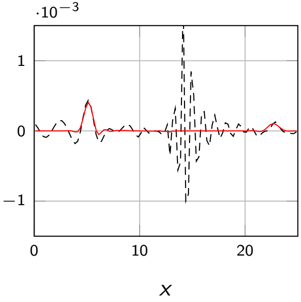
Validation: Subcritical flow and perturbation



(a) $t = 0$



(b) $t = 0.66$



(c) $t = 2$

Figure: Perturbation on a subcritical flow: $\eta - \eta^{eq}$

Table of contents

FEM+SUPG GF 2D high order

Acoustics

$$\partial_t u + \partial_x p = 0$$
$$\partial_t v + \partial_y p = 0$$
$$\partial_t p + \partial_x u + \partial_y v = 0$$

FEM+SUPG GF 2D high order

SUPG FEM for acoustics

$$\int \varphi(\partial_t u + \partial_x p) + \alpha\Delta\, \partial_x \varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha\Delta\, \partial_y \varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t p + \partial_x u + \partial_y v) + \alpha\Delta\, \partial_x \varphi(\partial_t u + \partial_x p) + \alpha\Delta\, \partial_y \varphi(\partial_t v + \partial_y p) = 0$$

Acoustics

$$\partial_t u + \partial_x p = 0$$
$$\partial_t v + \partial_y p = 0$$
$$\partial_t p + \partial_x u + \partial_y v = 0$$

FEM+SUPG GF 2D high order

SUPG FEM for acoustics

$$\int \varphi(\partial_t u + \partial_x p) + \alpha\Delta\, \partial_x\varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha\Delta\, \partial_y\varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t p + \partial_x u + \partial_y v) + \alpha\Delta\, \partial_x\varphi(\partial_t u + \partial_x p) + \alpha\Delta\, \partial_y\varphi(\partial_t v + \partial_y p) = 0$$

Acoustics

$$\begin{aligned} &\partial_t u + \partial_x p = 0\\ &\partial_t v + \partial_y p = 0\\ &\partial_t p + \partial_x u + \partial_y v = 0 \end{aligned}$$

Details on discretization

- Cartesian grid!!
- Gauss-Lobatto points for quadrature and Lagrange basis function
- Explicit arbitrary high order time discretization with Deferred Correction

Global flux in 2D

Main idea

Define

$$\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds \qquad \boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$$

So that

$$\partial_t p + \partial_x u + \partial_y v = \partial_t p + \partial_{xy}(\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0$$

Global flux in 2D

Main idea

Define

$$\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds \qquad \boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$$

So that

$$\partial_t p + \partial_x u + \partial_y v = \partial_t p + \partial_{xy}(\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0$$

Global Flux SUPG for acoustics

Define $\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds$ and $\boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$, with $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y \in V_h^K(\Omega_h)$, $\Phi := \boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y$.

Global flux in 2D

Main idea

Define

$$\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds \qquad \boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$$

So that

$$\partial_t p + \partial_x u + \partial_y v = \partial_t p + \partial_{xy}(\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0$$

Global Flux SUPG for acoustics

Define $\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds$ and $\boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$, with $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y \in V_h^K(\Omega_h)$, $\Phi := \boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y$.

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \Delta x\, \partial_x \varphi(\partial_t p + \partial_x \partial_y \Phi) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha \Delta y\, \partial_y \varphi(\partial_t p + \partial_x \partial_y \Phi) = 0$$

$$\int \varphi(\partial_t p + \partial_x \partial_y \Phi) + \alpha \Delta x\, \partial_x \varphi(\partial_t u + \partial_x p) + \alpha \Delta y\, \partial_y \varphi(\partial_t v + \partial_y p) = 0$$

Global Flux SUPG: changes

Global Flux SUPG for acoustics

$\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds$ and $\boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$, with $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y \in V_h^K(\Omega_h)$.

Changes in equilibrium

$$\begin{aligned} &\nabla \cdot \underline{v} = 0 \\ \Longrightarrow &\partial_x \partial_y (\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0 \\ \Longleftrightarrow &\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y = f(x) + g(y) \end{aligned}$$

Discrete equilibrium

$$\begin{aligned} &\partial_x \partial_y \Phi(x_i, y_j) = 0 \\ \Longrightarrow &\int_{x_0}^{x_i} \int_{y_0}^{y_j} \partial_y \partial_x \Phi(x,y) dxdy = 0 \quad \forall i,j \\ \Longrightarrow &\int_{x_0}^{x_i} \partial_x \Phi(x, y_j) dx - \int_{x_0}^{x_i} \partial_x \Phi(x, y_0) dx = 0 \quad \forall i,j \\ \Longrightarrow &\Phi(x_i, y_j) - \Phi(x_0, y_j) - \Phi(x_i, y_0) + \Phi(x_0, y_0) = 0 \quad \forall i,j \\ \Longrightarrow &\Phi(x_i, y_j) = f_i + g_j \end{aligned}$$

Myth buster

Global Flux is not global!

- In principle $\boldsymbol{\sigma}_x(x,y) = \int_{y_B}^{y} u(x,s)ds$ should be integrated from the beginning (bottom) of the domain y_B!
- In practice we always use $\partial_x\partial_y\boldsymbol{\sigma}_x(x,y)$ integrated in one cell!!!!
- So,

$$\boldsymbol{\sigma}_x(x,y) = \int_{y_B}^{y} u(x,s)ds = \underbrace{\int_{y_B}^{y_0} u(x,s)ds}_{\text{constant in one cell!}} + \int_{y_0}^{y} u(x,s)ds$$

  whatever constant we bring from outside the cell, is canceled out

$$\partial_y\boldsymbol{\sigma}_x(x,y) = \partial_y \int_{y_B}^{y} u(x,s)ds = \partial_y \int_{y_B}^{y_0} u(x,s)ds + \partial_y \int_{y_0}^{y} u(x,s)ds = \partial_y \int_{y_0}^{y} u(x,s)ds$$

- At the discrete level we have an integral operator I_y and a differential operator D_y that together give a weird averaging operator $D_y I_y$

Coriolis and sources

Extension to source terms

$$\partial_t u + \partial_x p = S_u$$
$$\partial_t v + \partial_y p = S_v$$
$$\partial_t p + \partial_x u + \partial_y v = S_p$$

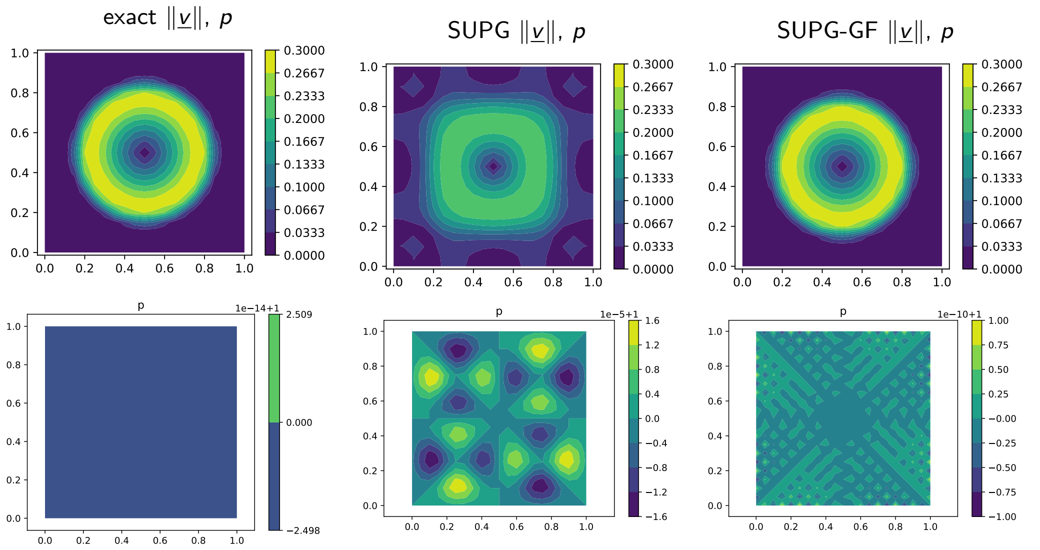
Source terms

- Coriolis
- Mass sources
- Friction

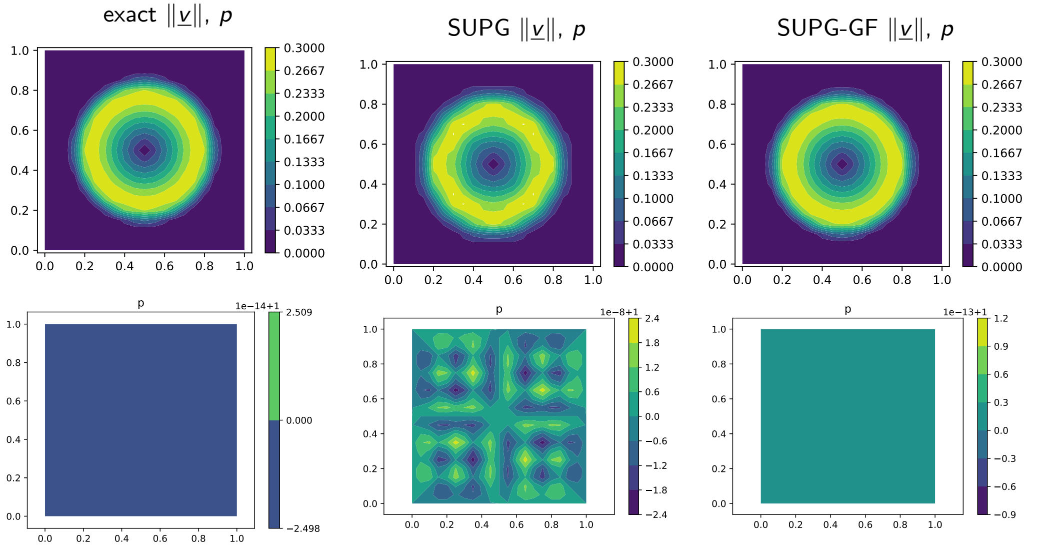
Global flux for sources

$$\mathcal{G}_u := p - \int^x S_u$$
$$\mathcal{G}_v := p - \int^y S_v$$
$$\mathcal{G}_p := \int^y u + \int^x v - \int^x \int^y S_u$$

Simulation of vortex (linear acoustics) FEM+SUPG: $\mathbb{Q}^1$, $N_x = N_y = 20$



Simulation of vortex (linear acoustics) FEM+SUPG: $\mathbb{Q}^2$, $N_x = N_y = 10$



Simulation of vortex: errors

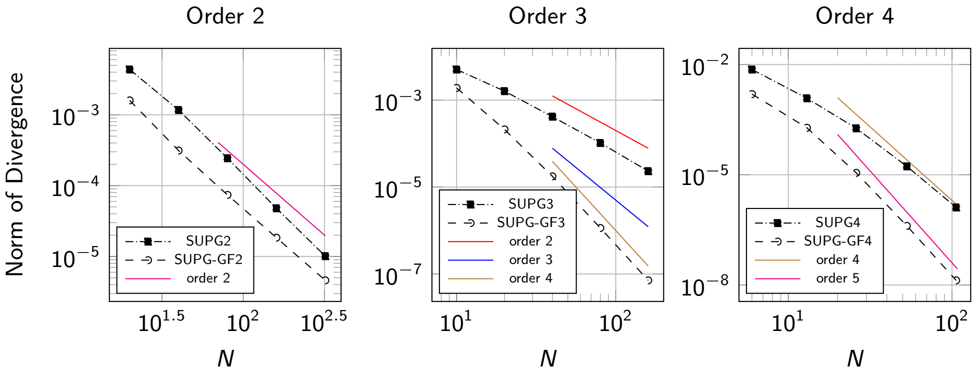


Figure: Smooth vortex: convergence of L^2 error of u with respect to the number of elements in x

Vortex simulation: divergence error

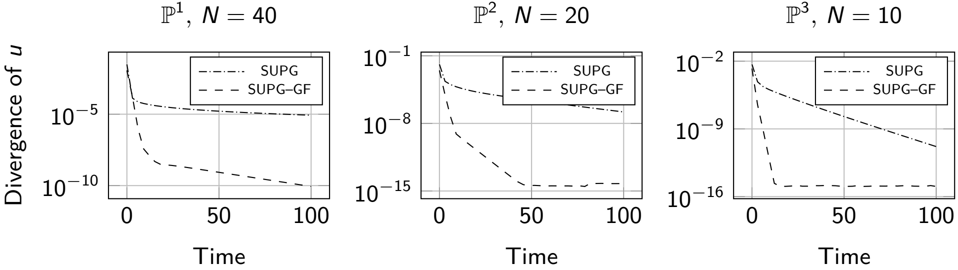


Figure: Norm of discrete divergence of $\underline{u}$ for SUPG ($\partial_x u + \partial_y v$) and SUPG–GF ($\partial_x \partial_y (\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y)$) simulations with respect to time for different orders

Vortex perturbation

Pressure perturbation

- Gaussian centered in $\underline{x}_p = (0.4, 0.43)$
- scaling coefficient $r_0 = 0.1$
- radius $\rho(\underline{x}) = \sqrt{||\underline{x} - \underline{x}_p||}/r_0$

$$\delta_p(\underline{x}) = \varepsilon e^{-\frac{1}{2(1-\rho(\underline{x}))^2}+\frac{1}{2}},$$

- final time $T = 0.35$

Vortex perturbation

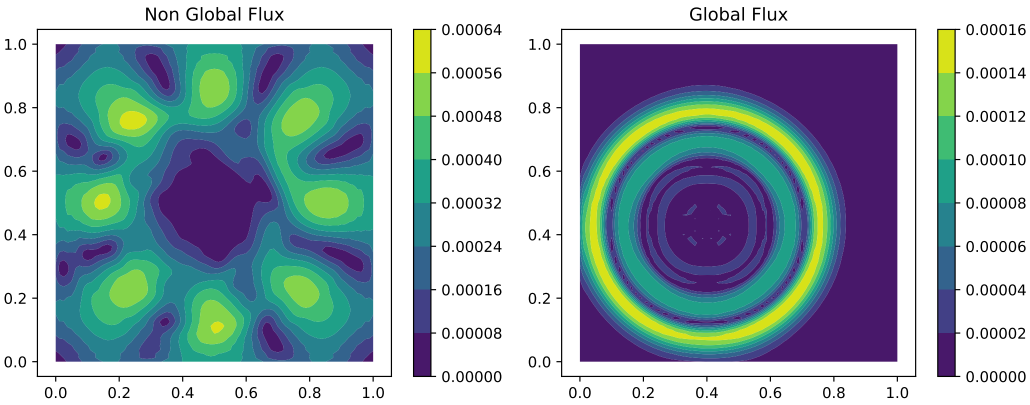


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^1$ with 80×80 cells and 6561 dofs.

Vortex perturbation

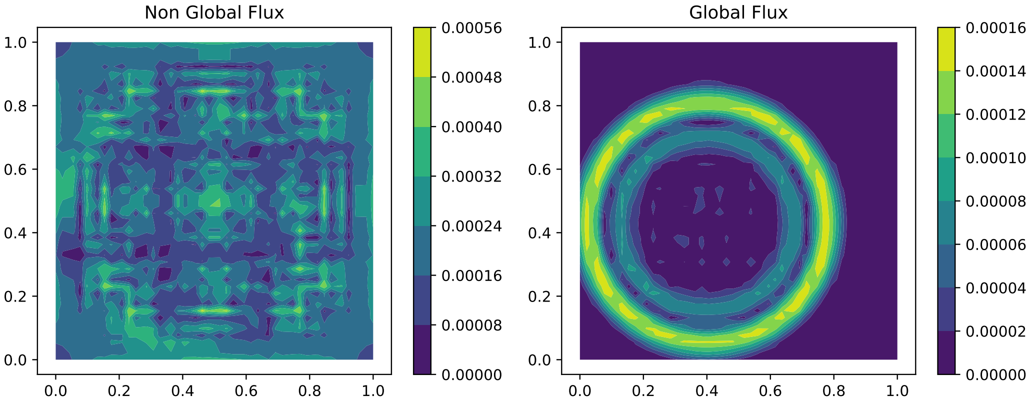


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 13×13 cells and 1600 dofs.

Vortex perturbation

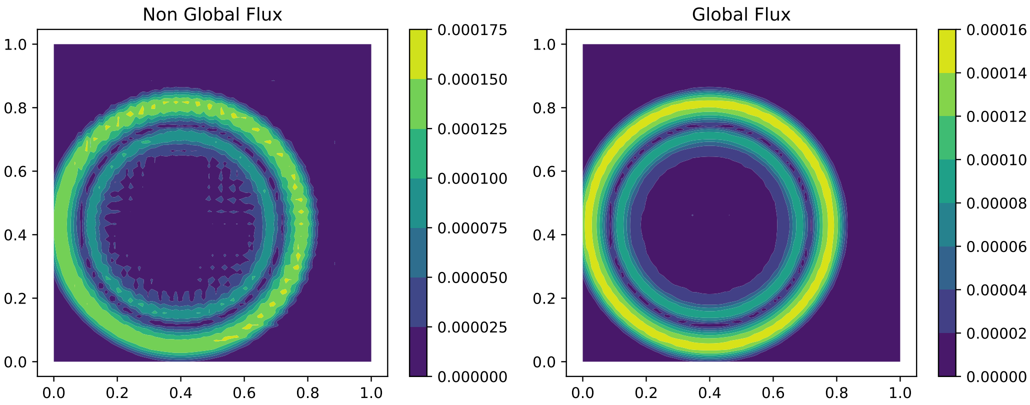


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 26 cells and 6241 dofs.

Table of contents

FEM+SUPG for GF

Discretizations

- GF+SUPG+FEM works easily also for nonlinear problems on paper
- GF+FV less trivial, because ...

GF+FEM+SUPG

$$\partial_t u + \partial_x F(u) + \partial_y G(u) = S(u) \Longrightarrow \partial_t u + \partial_x \partial_y \mathcal{G}(u) = 0$$

$$\mathcal{G}(u) := \int^y F(u) + \int^x G(u) - \int^x \int^y S(u);$$

$$\int_\Omega (\varphi + \alpha \Delta \partial_x \varphi J^x + \alpha \Delta \partial_y \varphi J^y)(\partial_t u + \partial_{xy} \mathcal{G}(u)) = 0 \qquad \forall \varphi.$$

FV for GF 2D

GF+FV

$$\partial_t u + \partial_x \partial_y \mathcal{G}(u) = 0$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (\partial_t u + \partial_{xy}\mathcal{G}(u)) \mathrm{d}x\mathrm{d}y = 0$$

$$\partial_t u_{ij} + \hat{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} - \hat{\mathcal{G}}_{i-\frac{1}{2},j+\frac{1}{2}} - \hat{\mathcal{G}}_{i+\frac{1}{2},j-\frac{1}{2}} + \hat{\mathcal{G}}_{i-\frac{1}{2},j-\frac{1}{2}} = 0$$

Corner numerical flux!!

- Upwind didn't work for nonlinear 2D problems (it worked in 1D, it works for 2D linear acoustics, but for nonlinear 2D all tentative methods were unstable)
- We ended up with the same SUPG scheme, applied on the dual mesh of the FV

FV for GF 2D

GF+FV

$$\partial_t u + \partial_x \partial_y \mathcal{G}(u) = 0$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (\partial_t u + \partial_{xy}\mathcal{G}(u)) \mathrm{d}x\mathrm{d}y = 0$$

$$\partial_t u_{ij} + \hat{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} - \hat{\mathcal{G}}_{i-\frac{1}{2},j+\frac{1}{2}} - \hat{\mathcal{G}}_{i+\frac{1}{2},j-\frac{1}{2}} + \hat{\mathcal{G}}_{i-\frac{1}{2},j-\frac{1}{2}} = 0$$

Corner numerical flux!!

- Upwind didn't work for nonlinear 2D problems (it worked in 1D, it works for 2D linear acoustics, but for nonlinear 2D all tentative methods were unstable)
- We ended up with the same SUPG scheme, applied on the dual mesh of the FV

Corner numerical flux: SUPG

$$\hat{\mathcal{G}}^{i+\ell,j+m}_{i+\frac{1}{2},j+\frac{1}{2}} = \overline{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} n^{(i+\ell,j+m)}_{i+\frac{1}{2},j+\frac{1}{2}} + \mathcal{D}^{(i+\ell,j+m)}_{i+\frac{1}{2},j+\frac{1}{2}},$$

$$\mathcal{D}^{(i+\ell,j+r)}_{i+\frac{1}{2},j+\frac{1}{2}} := \mathcal{D}(\widetilde{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}}, \bar{q}_{i+\frac{1}{2},j+\frac{1}{2}} | \mathbf{n}^{(i+\ell,j+r)}_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$= \alpha\Delta \int_{\widetilde{C}} \left(\frac{1}{\Delta x} J^x \partial_\xi \phi_{\ell,r} + \frac{1}{\Delta y} J^y \partial_\eta \phi_{\ell,r} \right) \partial_{\xi\eta} \widetilde{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} \mathrm{d}\xi \mathrm{d}\eta,$$

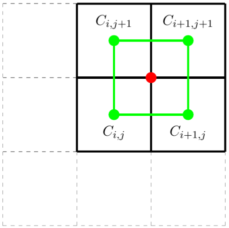


Table of contents

Euler equations: isentropic vortex (steady state)

IC

$$(\rho, u, v, p) = (1 + \delta\rho,\, \delta u,\, \delta v,\, 1 + \delta p)\,.$$

The test case is set up in a $[0,10] \times [0,10]$ domain with periodic boundary conditions and vortex radius $r = \sqrt{(x-5)^2 + (y-5)^2}$. The vortex strength is $\epsilon = 5$, and the entropy perturbation is assumed to be zero. Given these hypothesis, the perturbations on velocity and temperature can be written as

$$\begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = \frac{\epsilon}{2\pi} \exp\left(\frac{1-r^2}{2}\right) \begin{bmatrix} -(y-5) \\ (x-5) \end{bmatrix}, \qquad \delta T = -\frac{(\gamma-1)\epsilon^2}{8\gamma\pi^2} \exp(1-r^2).$$

It follows that the perturbations on density and pressure reads

$$\delta\rho = (1+\delta T)^{\frac{1}{\gamma-1}} - 1, \qquad \delta p = (1+\delta T)^{\frac{\gamma}{\gamma-1}} - 1.$$

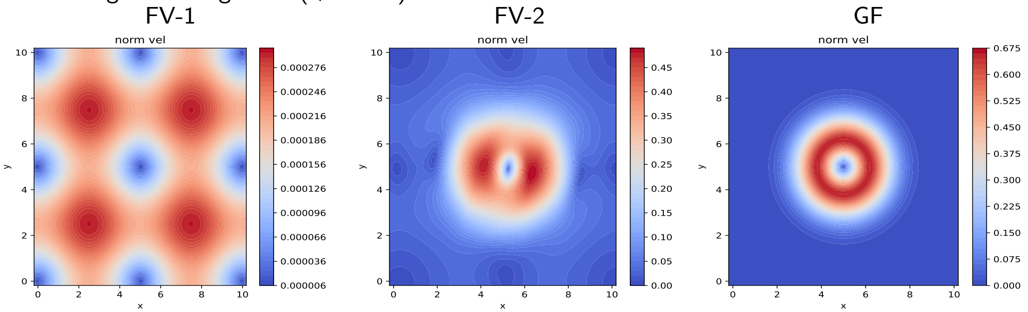
Euler equations: isentropic vortex (steady state)

Euler equations: isentropic vortex ($t_f = 1$). L_2 error and order of accuracy $\tilde{n}$ for FV-1, FV-2 and GF methods.

	ρ		ρu		ρv		ρE	
N_x, N_y	L_2	$\tilde{n}$	L_2	$\tilde{n}$	L_2	$\tilde{n}$	L_2	$\tilde{n}$
				FV-1				
20	3.58E-01	–	6.77E-01	–	6.77E-01	–	1.16E+00	–
40	2.47E-01	0.53	4.40E-01	0.62	4.40E-01	0.62	8.29E-01	0.48
80	1.49E-01	0.72	2.59E-01	0.76	2.59E-01	0.76	5.15E-01	0.68
160	8.33E-02	0.84	1.43E-01	0.85	1.43E-01	0.85	2.91E-01	0.82
320	4.42E-02	0.91	7.56E-02	0.91	7.56E-02	0.91	1.56E-01	0.90
				FV-2				
20	1.06E-01	–	2.05E-01	–	2.00E-01	–	4.32E-01	–
40	3.62E-02	1.55	6.74E-02	1.60	6.71E-02	1.57	1.20E-01	1.85
80	1.07E-02	1.76	1.93E-02	1.80	1.95E-02	1.78	2.91E-02	2.04
160	2.39E-03	2.16	5.58E-03	1.78	5.61E-03	1.79	7.04E-03	2.04
320	5.12E-04	2.22	1.39E-03	2.00	1.39E-03	2.01	1.56E-03	2.17
				GF				
20	1.52E-02	–	3.67E-02	–	3.67E-02	–	4.59E-02	–
40	5.95E-03	1.35	1.15E-02	1.67	1.15E-02	1.67	1.54E-02	1.57
80	1.76E-03	1.76	3.06E-03	1.90	3.06E-03	1.90	4.35E-03	1.82
160	4.69E-04	1.90	7.87E-04	1.96	7.87E-04	1.96	1.16E-03	1.90
320	1.21E-04	1.95	2.00E-04	1.97	2.00E-04	1.97	3.02E-04	1.94

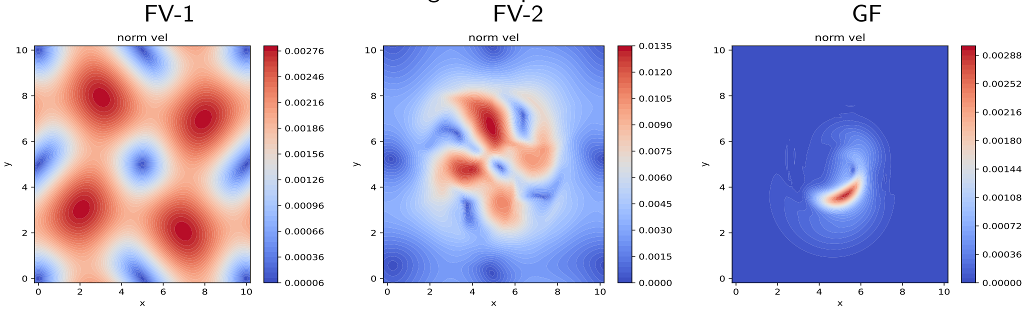
Euler equations: isentropic vortex (steady state)

Euler equations: isentropic vortex. Isocontours of the velocity norm obtained with FV-1, FV-2 and GF after a long time integration ($t_f = 200$)



Euler equations: isentropic vortex (steady state)

Euler equations: perturbation of the isentropic vortex. Isocontours of the $\rho - \rho_{eq}$ norm obtained with FV-1, FV-2 and GF at final time $t_f = 2$ with a 80×80 mesh. Take as IC the final simulation of longtime + perturbation



Euler equations: Kevin-Helmoltz instability

- Domain $[0,2] \times [-1/2, 1/2]$
- Final time $t_f = 80$
- initial condition

$$\rho = \gamma + \mathcal{H}(y)\, r, \qquad u = M\,\mathcal{H}(y), \qquad v = \delta\, M \sin(2\pi x), \qquad p = 1,$$

- Mach number parameter $M = 10^{-2}$, $r = 10^{-3}$, $\delta = 0.1$
- $\mathcal{H}(y)$

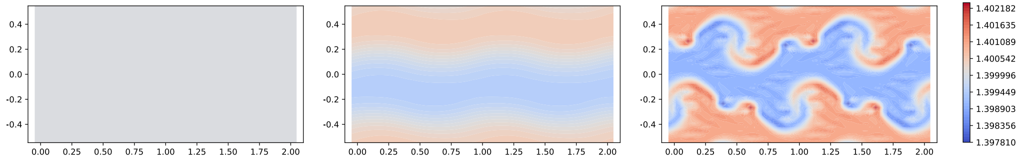
$$\mathcal{H}(y) = \begin{cases} -\sin\left(\frac{\pi}{\omega}\left(y + \frac{1}{4}\right)\right), & \text{if} \quad -\frac{1}{4} - \frac{\omega}{2} \leq y < -\frac{1}{4} + \frac{\omega}{2}, \\ -1, & \text{if} \quad -\frac{1}{4} + \frac{\omega}{2} \leq y < \frac{1}{4} - \frac{\omega}{2}, \\ \sin\left(\frac{\pi}{\omega}\left(y - \frac{1}{4}\right)\right), & \text{if} \quad \frac{1}{4} - \frac{\omega}{2} \leq y < \frac{1}{4} + \frac{\omega}{2}, \\ 1 & \text{else}, \end{cases}$$

where $\omega = 1/16$.

Euler equations: Kevin-Helmoltz instability

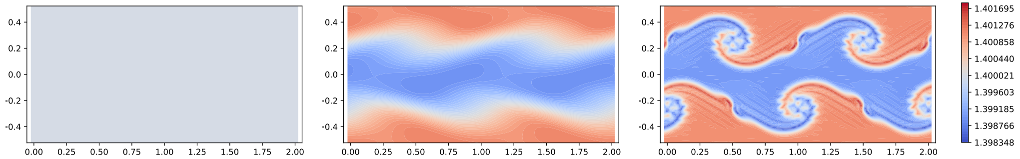
$N_x = 64,\ N_y = 32$

FV-1 FV-2 GF

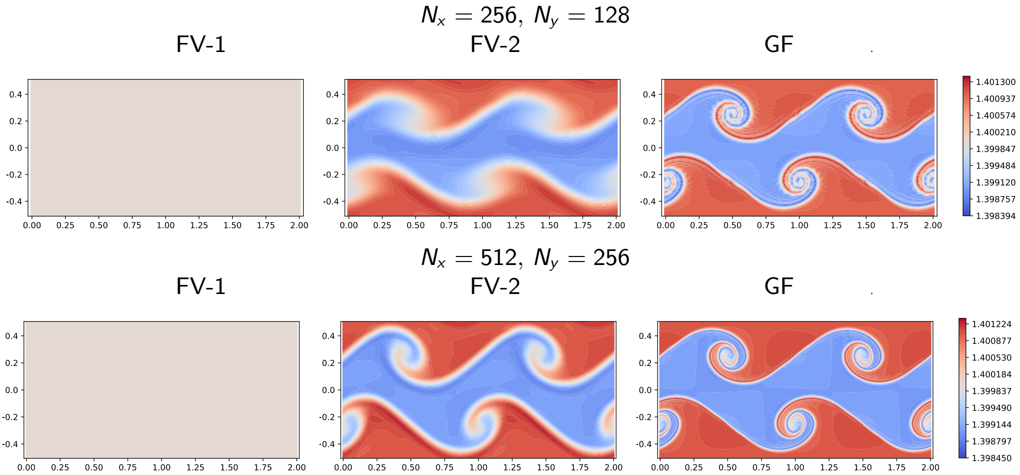


$N_x = 128,\ N_y = 64$

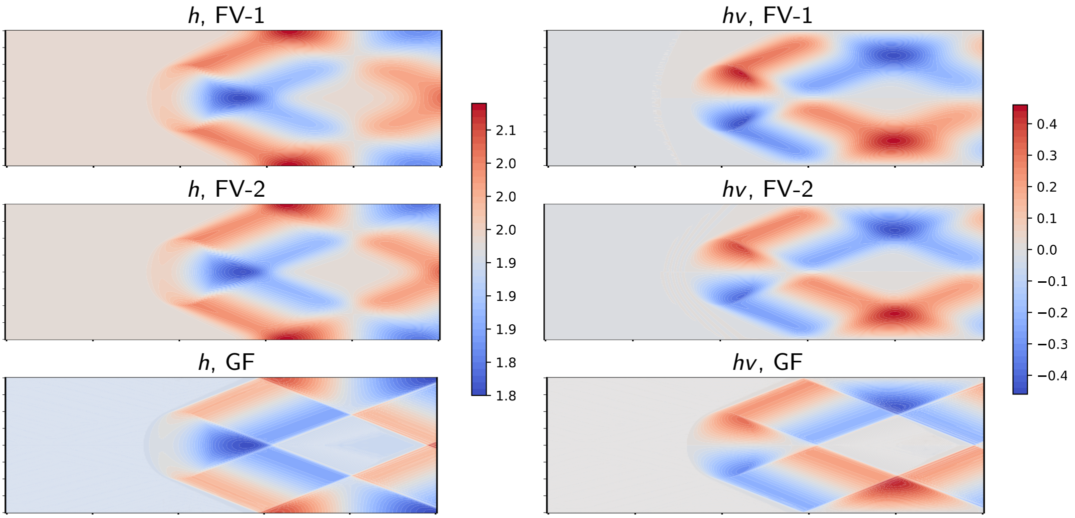
FV-1 FV-2 GF



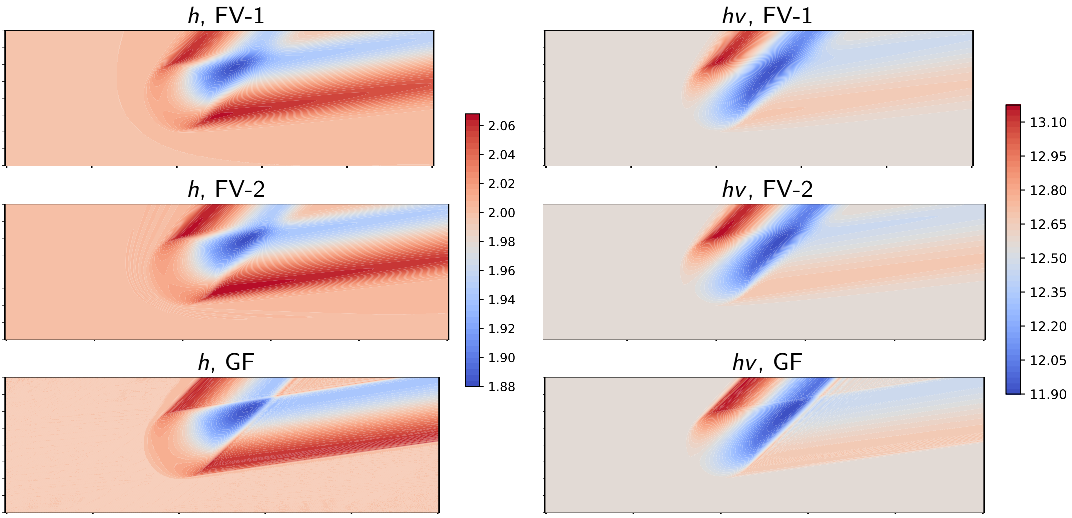
Euler equations: Kevin-Helmoltz instability



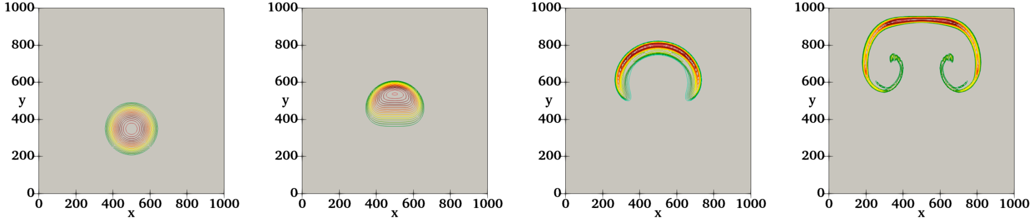
Shallow water: subcritical flow with bathymetry



Shallow water: subcritical flow with bathymetry



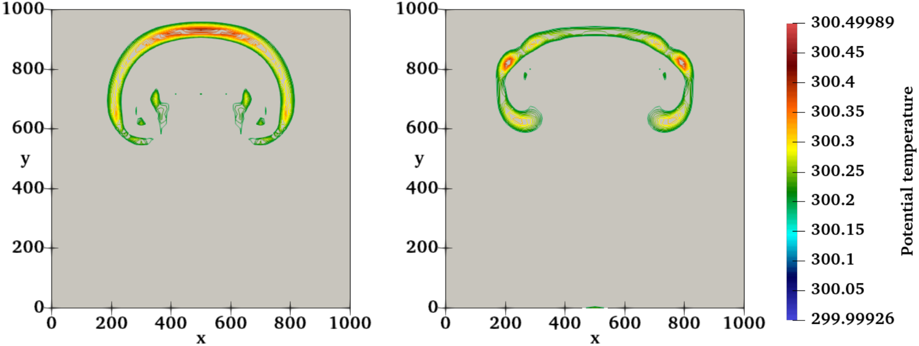
Euler equations, FEM: Thermal rising bubble



Top: GF 150x150

Bottom left: GF 60x60

Bottom right: SUPG 60x60



Euler equations, FV: Low-Mach Shu Vortex

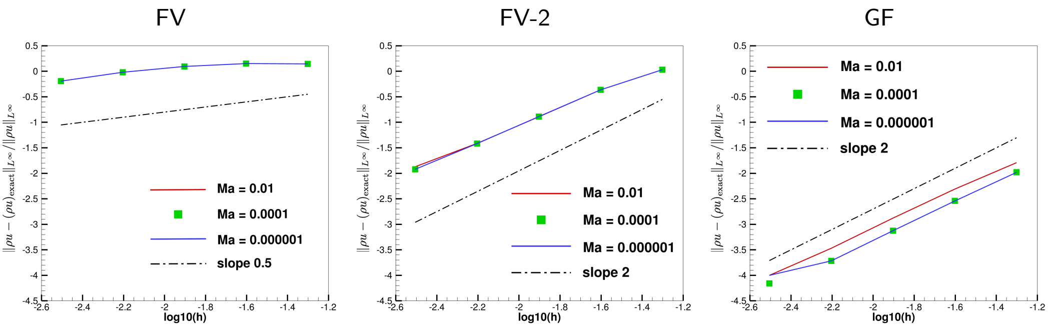


Table of contents

Extensions and Perspectives

Summary

- Global Flux to preserve moving equilibria
- 1D integrate the source and unique flux
- 2D integrate F in y and G in x
- Some superconvergence in steady states
- Extra accuracy in vorticity like problems
- Extra accuracy in low Mach problems
- Small stability issues with very very long time simulations in nonlinear 2D
- No problem with shocks (we were surprised)

Perspectives

- Fix the long time behavior for vortices
- Other methods: DG seems less trivial
- Other geometries: Immersed Boundaries
- Riemann solver for corner problems?
- Non Cartesian meshes

Extensions and Perspectives

Summary

- Global Flux to preserve moving equilibria
- 1D integrate the source and unique flux
- 2D integrate F in y and G in x
- Some superconvergence in steady states
- Extra accuracy in vorticity like problems
- Extra accuracy in low Mach problems
- Small stability issues with very very long time simulations in nonlinear 2D
- No problem with shocks (we were surprised)

Perspectives

- Fix the long time behavior for vortices
- Other methods: DG seems less trivial
- Other geometries: Immersed Boundaries
- Riemann solver for corner problems?
- Non Cartesian meshes

THANKS!!

State of the art techniques (part 1)

Subtract equilibrium

- Know analytical equilibrium
- Dedner 2004[a] and Berberich 2021[b]

[a] Dedner, A., Rohde, C., Schupp, B., & Wesenberg, M. (2004). Computing and Visualization in Science, 7(2), 79-96.

[b] J. P. Berberich, P. Chandrashekar, and C. Klingenberg. Computers & Fluids, 219:104858, 2021.

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq} := (h^{eq}, u^{eq}, v^{eq})$
- Discrete exact equilibrium residual: $\mathcal{S}^{eq}(t^n) := \mathcal{S}(V^{eq}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq}(t^n)$

State of the art techniques (part 1)

Subtract equilibrium

- Know analytical equilibrium
- Dedner 2004[a] and Berberich 2021[b]

[a] Dedner, A., Rohde, C., Schupp, B., & Wesenberg, M. (2004). Computing and Visualization in Science, 7(2), 79-96.

[b] J. P. Berberich, P. Chandrashekar, and C. Klingenberg. Computers & Fluids, 219:104858, 2021.

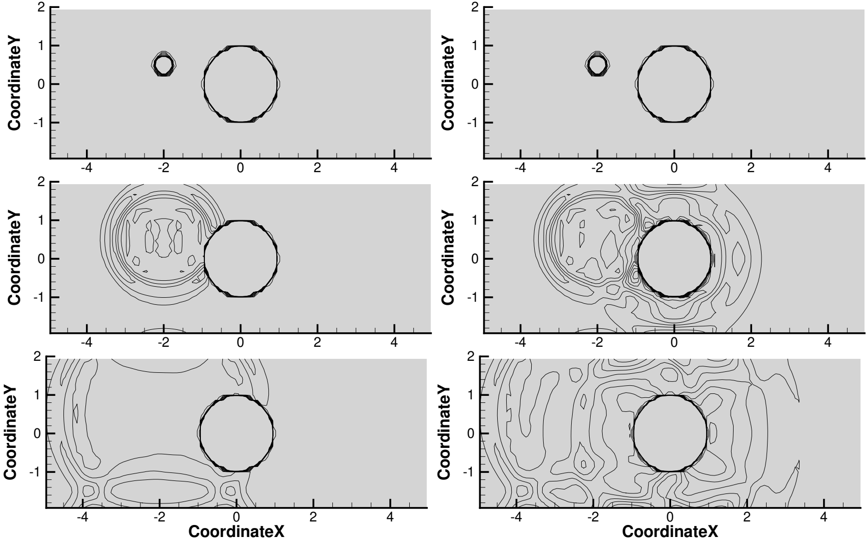
Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq} := (h^{eq}, u^{eq}, v^{eq})$
- Discrete exact equilibrium residual: $\mathcal{S}^{eq}(t^n) := \mathcal{S}(V^{eq}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq}(t^n)$

Properties

- ☺ Ridiculously well balanced: $V^n = V^{eq} \Longrightarrow V^{n+1} = V^{eq}$
- ☹ Know equlibrium a priori
- ☺ Lake at rest
- ☹ Stationary waves
- ☹ 2D vortices

Example: subtract equilibrium[2]



[2] Ciallella, M., Micalizzi, L., Öffner, P., & Torlo, D. (2022). Computers & Fluids, 247, 105630.

State of the art techniques (part 2)[3]

Equilibrium reconstruction

- In every cell solve an ODE at reconstruction/quadrature points, constrained with the state V^n (BVP)
- ODE solver either exact or very accurate
- Malaga school

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq,ODE}$:=ODE_Solver(1) subject to V^n
- Discrete equilibrium residual: $\mathcal{S}^{eq,ODE}(t^n) := \mathcal{S}(V^{eq,ODE}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq,ODE}(t^n)$

[3] Castro, M. J., & Parés, C. (2020). Journal of Scientific Computing, 82(2), 48.

State of the art techniques (part 2)[3]

Equilibrium reconstruction

- In every cell solve an ODE at reconstruction/quadrature points, constrained with the state V^n (BVP)
- ODE solver either exact or very accurate
- Malaga school

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq,ODE}$:=ODE_Solver(1) subject to V^n
- Discrete equilibrium residual: $\mathcal{S}^{eq,ODE}(t^n) := \mathcal{S}(V^{eq,ODE}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq,ODE}(t^n)$

Properties

☺ Exactly well-balanced $V^n = V^{eq,ODE} \implies V^{n+1} = V^{eq,ODE}$

☺ For all equilibria of one type

☹ Expensive (ODE solver for each cell)

☺ Lake at rest

☺ Stationary waves

☹ Problem for transcritical flows $u = \sqrt{gh}$

☹ 2D vortices

[3] Castro, M. J., & Parés, C. (2020). Journal of Scientific Computing, 82(2), 48.

State of the art techniques (part 3)[4]

Riemann problem modification

- For FV schemes
- Change the Riemann problem approximation
- Exploit (1) such that at equilibrium it is satisfied by the Riemann problem
- Michel-Dansac 2016

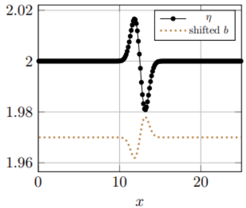
Properties

- Exactly well-balanced (if (1) analytically invertible else accurate solver) $V^n = V^{eq,ODE} \implies V^{n+1} = V^{eq,ODE}$
- ☺ For all equilibria of one type
- ☹ Computations by hand for Riemann Solver
- ☹ Only 1st order, blending with high order
- ☺ Lake at rest
- ☺ Stationary waves
- ☹ Problem for transcritical flows $u = \sqrt{gh}$
- ☹ 2D vortices

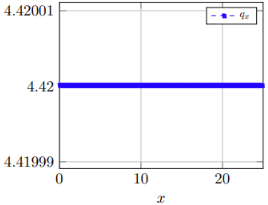
[4] Michel-Dansac, V., Berthon, C., Clain, S., & Foucher, F. (2016). Computers & Mathematics with Applications, 72(3), 568-593.

Example: Riemann Problem Change[5]

SUBCRITICAL

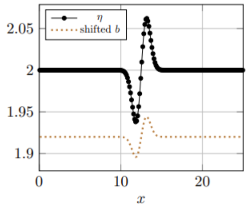


(a) free surface η and bathymetry b, shifted and rescaled

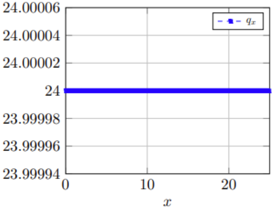


(b) discharge q_x

SUPERCRITICAL



(a) free surface η and bathymetry b, shifted and rescaled

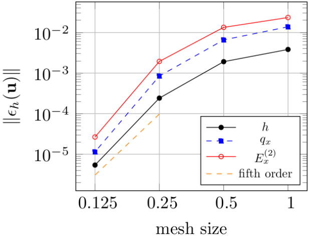


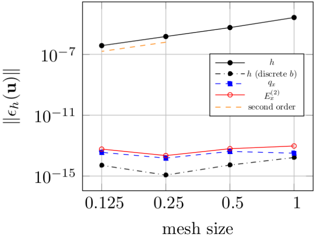
(b) discharge q_x

[5] Ciallella, M., Micalizzi, L., Michel-Dansac, V., Öffner, P., & Torlo, D. (2025)

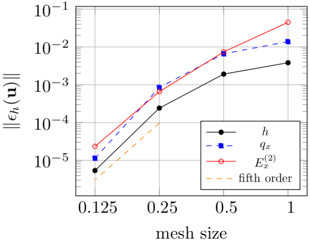
Example: Riemann Problem Change[5]

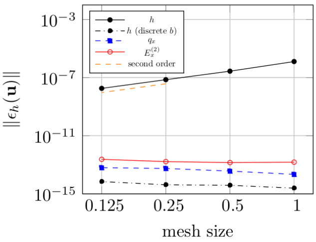
SUBCRITICAL





SUPERCRITICAL





[5] Ciallella, M., Micalizzi, L., Michel-Dansac, V., Öffner, P., & Torlo, D. (2025)