Model order reduction for Friedrichs' systems: a bridge between elliptic and hyperbolic problems

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Motivation

Friedrichs' Systems 1958[a]

- A model for transonic flows
- Both **elliptic** and **hyperbolic** regimes
- One **hyperbolic** model

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Unify physics

- Historical effort to find one general model
- Still research on unification of models
  - A **hyperbolic** model for viscous Newtonian flows[a]: Euler, Navier-Stokes, non Newtonian fluids, solid deformation.[b]
- **Hyperbolic** equations generalize many other models
  - General relativity [c]

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Hyperbolic model

- First order derivatives
- Extra variables (with or without physical meaning)
- Preserve **causality principle** and **finite speed propagation**[b]

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Friedrichs' System in Brief and Goals

What are Friedrichs' systems?

- **Hyperbolic** systems (first order PDEs)
- Also elliptic $\Longrightarrow$ extra variables
- **Linear** systems
- Admissible **boundary conditions**: technical, but natural and elegant

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What can we solve with Friedrichs' Systems?

- Linear systems (today)
  - Wave equations
  - Diffusion equations
  - Advection-diffusion-reaction equations
  - Div-grad problems
  - Linear elasticity
  - Curl-curl problems as Maxwell's equations
  - Mangeto-hydrodynamics
  - Klein-Gordon

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What we want to do

- Use Discontinuous Galerkin (**DG**) to solve FS
- Exploit linearity of the problem to ROM
- Simple **Galerkin projection** for ROM
- *A posteriori* **error estimator** for Greedy

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What we do **NOT** treat in this talk

- **Nonlinear** problems (in future linearized Euler with hyper reduction)
- Advection dominated problems and slow **Kolmogorov N-width** decay (in future with calibration techniques)

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Friedrichs' Systems[1]

$\Omega \subset \mathbb{R}^d$ geometrical domain

d geometrical dimension,

m system dimension,

$\mathcal{A}^k \in [L^\infty(\Omega)]^{m\times m}$, $k = 0, \dots, d$,

$\mathcal{A}^k = (\mathcal{A}^k)^T$ a.e. in Ω, $k = 1, \dots, d$,

$$\mathcal{A}^0 + (\mathcal{A}^0)^T - \sum_{k=1}^{d} \partial_k \mathcal{A}^k \geq 2\mu_0 \mathbb{I}_m > 0.$$

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$\mathbb{R}^m = \ker(\mathcal{D} - \mathcal{M}) + \ker(\mathcal{D} + \mathcal{M})$.

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Friedrichs' System

$$\begin{cases} Az = f, & \text{in } \Omega, \\ (\mathcal{D} - \mathcal{M})z = 0, & \text{on } \partial\Omega. \end{cases}$$

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Example: Advection-Diffusion-Reaction

$$-\nabla \cdot (\kappa \nabla u) + \beta \cdot \nabla u + \mu u = r$$

$$\mu - \frac{1}{2}\nabla \cdot \beta \geq \mu_0 > 0, \qquad \kappa \in [L^\infty]^{d\times d}, \kappa \geq \kappa_0 \mathbb{I}_d > 0.$$

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Introduce an extra variable $v = -\kappa \nabla u$, so that $z = \begin{pmatrix} v \\ u \end{pmatrix}$ is our unknown.

$$\mathcal{A}^0 = \begin{bmatrix} \kappa^{-1} & 0_{d,1} \\ 0_{1,d} & \mu \end{bmatrix}, \qquad \mathcal{A}^k = \begin{bmatrix} 0_{d,d} & e_k \\ (e_k)^T & \beta_k \end{bmatrix}, \qquad f = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

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$$Az = f \iff \mathcal{A}^0 z + \sum_{k=1}^{d} \mathcal{A}^k \partial_k z = f \iff \begin{cases} \kappa^{-1} v + \nabla u = 0 \\ \mu u + \nabla \cdot v + \beta \cdot \nabla u = r \end{cases}$$

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Boundary operator

$$\mathcal{D} = \sum_{k=1}^{d} n_k \mathcal{A}^k = \begin{bmatrix} 0_{d,d} & n \\ n^t & \beta \cdot n \end{bmatrix}$$

Dirichlet homogeneous

$$\mathcal{M} := \begin{bmatrix} 0_{d,d} & -n \\ n^t & 0 \end{bmatrix}$$

Robin/Neumann: $v \cdot n = \gamma u$

$$\mathcal{M} := \begin{bmatrix} 0_{d,d} & n \\ -n^t & 2\gamma + \beta \cdot n \end{bmatrix}$$

with $\gamma = (\beta \cdot n)^-$ inflow/outflow.

Example: Linear Elasticity

$$\begin{cases} \sigma - \frac{1}{d+\lambda}\mathrm{tr}(\sigma)\mathbb{I}_{d,d} - \frac{1}{2}(\nabla u + (\nabla u)^T) = 0, \\ -\frac{1}{2}\nabla \cdot (\sigma + \sigma^T) + \alpha u = r, \end{cases}$$

$\alpha, \lambda > 0$ $\quad$ ($\lambda \sim$ compressibility, α for hypotesis, can be relaxed).

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Computing the trace of the first equation

$$\sigma = \frac{1}{2}(\nabla u + (\nabla u)^T) + \lambda^{-1}(\nabla \cdot u)\mathbb{I}_{d,d}.$$

$$\mathcal{A}^0 = \begin{bmatrix} \mathbb{I}_{d^2,d^2} - \frac{1}{d+\lambda}\mathcal{Z} & 0_{d^2,d} \\ 0_{d,d^2} & \alpha\mathbb{I}_{d,d} \end{bmatrix}, \; \mathcal{A}^k = \begin{bmatrix} 0_{d^2,d^2} & \mathcal{E}^k \\ (\mathcal{E}^k)^T & 0_{d,d} \end{bmatrix}, \; f = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

$$\mathcal{Z}_{[ij],[kl]} = \delta_{ij}\delta_{kl}, \quad \mathcal{E}^k_{[ij],l} = -\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

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$$\mathcal{D} = \sum_{k=1}^{d} n_k \mathcal{A}^k = \begin{bmatrix} 0_{d^2,d^2} & \mathcal{N} \\ \mathcal{N}^T & 0_{d,d} \end{bmatrix}$$

$$\mathcal{N}\xi := -\frac{1}{2}(n \otimes \xi + \xi \otimes n)$$

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Example: Curl-curl (steady Maxwell's equations)

$$\begin{cases} \mu H + \nabla \times E = r, \\ \sigma E - \nabla \times H = g, \end{cases}$$

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$$\mathcal{R}^k_{ij} = \epsilon_{ikj} \text{ (Levi-Civita tensor)}$$

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$$\mathcal{T}\xi := n \times \xi$$

Homogeneous Dirichlet on tangential of electric field

$$\mathcal{M} = \begin{bmatrix} 0_{d,d} & -\mathcal{T} \\ \mathcal{T}^T & 0_{d,d} \end{bmatrix}$$

Homogeneous Dirichlet on tangential of magnetic field

$$\mathcal{M} = \begin{bmatrix} 0_{d,d} & \mathcal{T} \\ -\mathcal{T}^T & 0_{d,d} \end{bmatrix}$$

Friedrichs' Systems Properties

Well posedness[a][b]

We have to introduce

- Operator A and its formal dual $\widetilde{A}$
- Extension to **graph space** with forms a and a^*
- D from A, $\widetilde{A}$
- **Cone formalism** to include BC

Theorem

Given $(A, \tilde{A})$, D, M, a, a^* + *hypotheses, then the problem*

$$a(z, y) = (f, y) \forall y \in V$$

is ***well-posed*** *and the solution satisfy the strong form in the weak sense.*

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Why interested in Friedrichs' Systems?

- One model $\Longrightarrow$ many models
- One method $\Longrightarrow$ tons of models
- Symmetry matrices
  - structure preserving (e.g. entropy solutions for Euler)
  - Easier to be solved
- Existence of ultraweak form + Conformal methods $\Longrightarrow$ isometry, error = residual
- Error estimates also for hyperbolic
- Large(r) matrices $\Longrightarrow$ ROM

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DG discretization

- $V_h := [\mathbb{P}^p_d(\mathcal{T}_h)]^m$

$a_h(z, y_h) :=$

$$\sum_{T\in\mathcal{T}_h} (Az, y_h)_{L^2(T)} + \frac{1}{2}\sum_{F\in\mathcal{F}^b} ((\mathcal{M}-\mathcal{D})z, y_h)_{L^2(F)}$$

$$-\sum_{F\in\mathcal{F}^i_h} (\mathcal{D}_F[\![z]\!], \{\{y_h\}\})_{L^2(F)}$$

$$+\sum_{F\in\mathcal{F}^b_h} (S^b_F z, y_h)_{L^2(F)} + \sum_{F\in\mathcal{F}^i_h} (S^i_h[\![z]\!], [\![y_h]\!])_{L^2(F)}$$

DG for Friedrichs' Systems

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$$- \sum_{F\in\mathcal{F}^i_h} (\mathcal{D}_F[\![z]\!], \{\{y_h\}\})_{L^2(F)}$$

$$+ \sum_{F\in\mathcal{F}^b_h} (S^b_F z, y_h)_{L^2(F)} + \sum_{F\in\mathcal{F}^i_h} (S^i_h[\![z]\!], [\![y_h]\!])_{L^2(F)}$$

a_h properties

- Consistent, coercive
- Inf-sup stability

$$|||z_h||| \lesssim \sup_{y_h\neq 0} \frac{a_h(z_h, y_h)}{|||y_h|||}$$

$$|||y|||^2 := ||y||^2_{L^2} + |y|^2_M + |y|^2_S + \sum_{T\in\mathcal{T}_h} h_T \left\| A^k \partial_k y \right\|_{L^2(T)}$$

- Boundness $a_h(w, y_h) \lesssim |||w|||_* |||y_h|||$ with

$$|||y|||^2_* = |||y|||^2 + \sum_{T\in\mathcal{T}_h} (h_T^{-1} ||y||^2_{L^2(T)} + ||y||^2_{L^2(\partial T)})$$

Error estimate $\quad |||z - z_h||| \leq C_z h^{k+1/2}$

DG for Friedrichs' Systems

DG discretization

- Stable
- High Order Accurate
- Linear discretization of the linear operators
- Easy parallel implementation of the system assembly
- Affine decomposition easy to exploit
- Implemented in `deal.II`[2]

[2] `dealii.org`

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Galerkin Projection for Friedrichs' Systems FOM

ROM discretization of parametric Friedrichs' system

- FOM problem: $a_h(z, y_h; \mu) = (f(\mu), y_h)_{L^2} \qquad \forall y_h \in V_h$
- $V_{RB} \Longleftarrow$ POD or Greedy
- Projection on V_{RB}: $z_{RB} = \sum_{j=1}^{N_{RB}} z_{RB}^j \psi_j^{RB}$ with $\{\psi_j^{RB}\}_{j=1}^{N_{RB}}$ a basis of V_{RB}

$$a_h(z_{RB}, \psi_j^{RB}; \mu) = (f(\mu), \psi_j^{RB})_{L^2} \qquad \text{for all } j = 1, \dots, N_{RB}.$$

- Parametric affine decomposition:

$$a_h(z, y; \mu) = \sum_{\ell=1}^{N_{affine}} \theta_\ell(\mu) a_{\ell,h}(z, y), \qquad (f(\mu), y) = \sum_{\ell=1}^{N^f_{affine}} \theta^f_\ell(\mu)(f_\ell, y),$$

- Reduced problem: find z_{RB}^j such that

$$\sum_{i=1}^{N_{RB}} z_{RB}^i(\mu) \sum_{\ell=1}^{N_{affine}} \theta_\ell(\mu) a_{\ell,h}(\psi_i^{RB}, \psi_j^{RB}) = \sum_{\ell=1}^{N^f_{affine}} \theta^f_\ell(\mu)(f_\ell, \psi_j^{RB}) \qquad \text{for all } j = 1, \dots, N_{RB}.$$

A posteriori error estimator in L^2

Error bound

- $e_h(\mu) := z_h(\mu) - z_{RB}(\mu)$, Riesz residual representing vector $\hat{r}^{\mu}_{RB} = F - Az_{RB}$ in L^2, coercivity

$$||e_h(\mu)||_2 \leq \frac{||\hat{r}^{\mu}_{RB}||_2}{\mu_0} =: \Delta_{2,RB}(\mu)$$

Effectiveness error estimator

Boundness of a_h depends on norm $|||\cdot|||_*$, hence, very bad estimates (we are working on that)

Cheap computation of the error estimator

$$\hat{r}^{\mu}_{RB} := F - Az_{RB} = \sum_{\ell=1}^{N^f_{affine}} \theta^f_{\ell}(\mu) F_{\ell} - \sum_{k=1}^{N_{affine}} \theta_k(\mu) A_k \left(\sum_{n=1}^{N_{RB}} z^n_{RB} \psi^{RB}_n \right)$$

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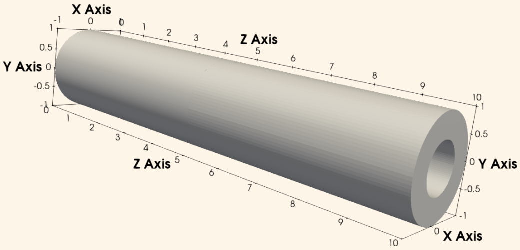
Compressible Linear Elasticity: Problem

$$\begin{cases} \sigma - \frac{1}{d+\lambda}\mathrm{tr}(\sigma)\mathbb{I}_{d,d} - \frac{1}{2}(\nabla u + (\nabla u)^T) = 0, \\ -\frac{1}{2}\nabla \cdot (\sigma + \sigma^T) + \alpha u = r, \end{cases}$$

$$\boldsymbol{\mu} = (\alpha, \lambda) \in ([100, 1000], [100, 1000])$$
$$r^T = (0, -1, 0)$$

Domain Ω a cylindrical shell along the z-axis with inner radius 1, outer radius 3 and height 10



Boundary Conditions

$$\Gamma_D = \partial\Omega \cap \{z = 0\} \qquad \Gamma_N = \partial\Omega \setminus \Gamma_D$$

Mixed boundary conditions:

$$\langle \mathcal{M}(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)\rangle_{V',V} =$$
$$\langle \boldsymbol{\sigma} \cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_D} - \langle \boldsymbol{\tau} \cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_D}$$
$$- \langle \boldsymbol{\sigma} \cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N} + \langle \boldsymbol{\tau} \cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_N}$$

Compressible Linear Elasticity: Error Decay

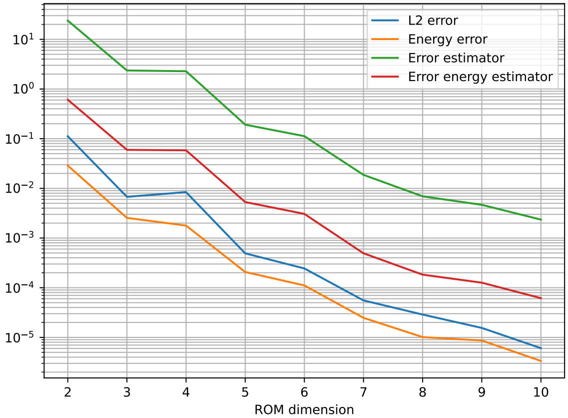


Figure: Error decay for different number of reduced basis N_{RB} (each component) for $||\cdot||_2$ and $||\cdot||_\mu = a_h(\cdot,\cdot;\mu)$ and their estimators

Compressible Linear Elasticity: Speed-up

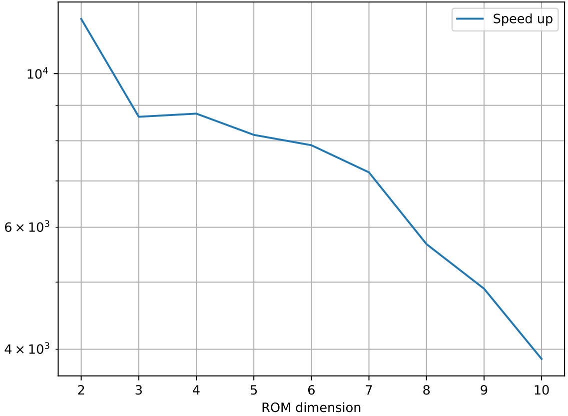


Figure: Speed up factor for different number of reduced basis N_{RB}

Compressible Linear Elasticity: Exploring parameter space

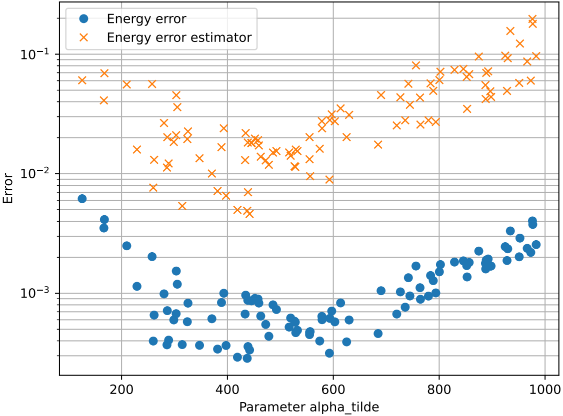


Figure: Error in energy norm and its estimator on a test set with respect to λ for $N_{RB} = 4$

Compressible Linear Elasticity: Simulations

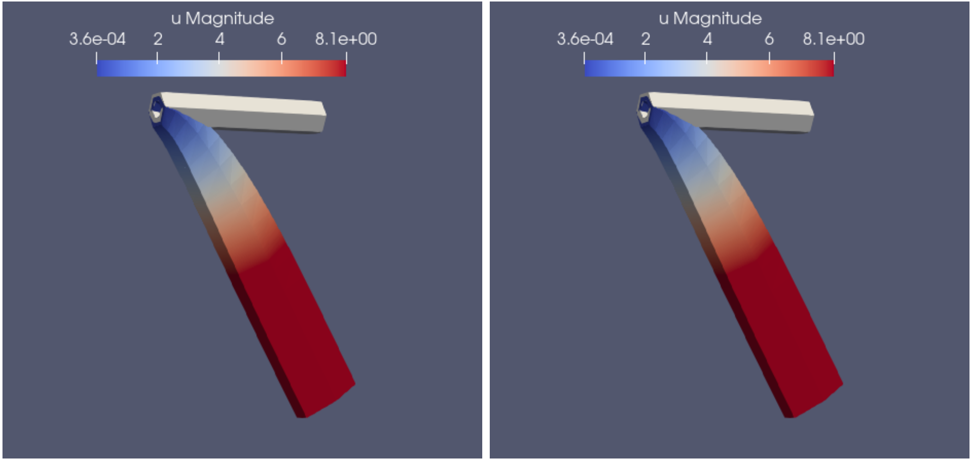


Figure: Simulation for $\alpha = 888.9$, $\lambda = 839.8$, FOM left ROM ($N_{RB} = 4$) right

Compressible Linear Elasticity: Simulations

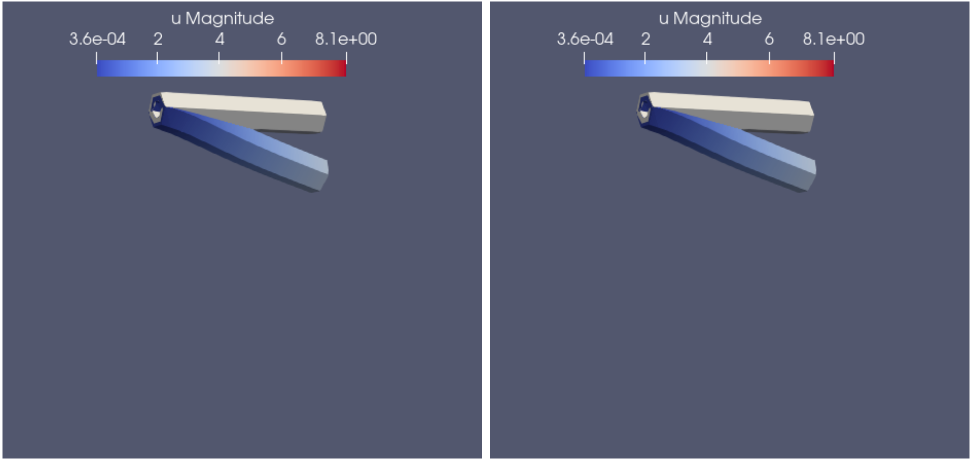


Figure: Simulation for $\alpha = 324.3$, $\lambda = 878.4$, FOM left ROM ($N_{RB} = 4$) right

Curl-Curl (Maxwell): Problem

Maxwell equations in the stationary regime
Ω is a **torus** with radii $r = 0.5$ and $R = 2$

$$\begin{pmatrix} \mu H + \nabla \times E \\ \sigma E - \nabla \times H \end{pmatrix} = \begin{pmatrix} r \\ g \end{pmatrix}, \quad \forall \mathbf{x} \in \Omega,$$

$$\mu \in [0.5, 2], \qquad \sigma \in [0.5, 3].$$

Boundary conditions

$$n \times E = \mathbf{0} \qquad \mathcal{M} = \begin{bmatrix} 0 & \mathcal{T} \\ \mathcal{T}^T & 0 \end{bmatrix}$$

$$\mathcal{T}\xi = n \times \xi, \qquad \forall \xi \in \mathbb{R}^3.$$

$$H_{\text{exact}}(\mathbf{x}) = -\frac{1}{\mu}\left(\frac{2xy}{\sqrt{x^2+z^2}}, \frac{-4y^2\sqrt{x^2+z^2} + \sqrt{x^2+z^2}(-12(x^2+z^2)-15) + 32(x^2+z^2)}{4(x^2+z^2)}, \frac{2xy}{\sqrt{x^2+z^2}}\right),$$

$$E_{\text{exact}}(\mathbf{x}) = \left(\frac{z}{\sqrt{x^2+z^2}}, 0, -\frac{x}{\sqrt{x^2+z^2}}\right) \cdot \left(r^2 - y^2 - \left(R - \sqrt{x^2+z^2}\right)^2\right),$$

The source terms are defined consequently as

$$r(\mathbf{x}) = 0, \qquad g(\mathbf{x}) = \sigma E_{\text{exact}} - \nabla \times H_{\text{exact}}. \tag{1}$$

Curl-Curl (Maxwell): Problem

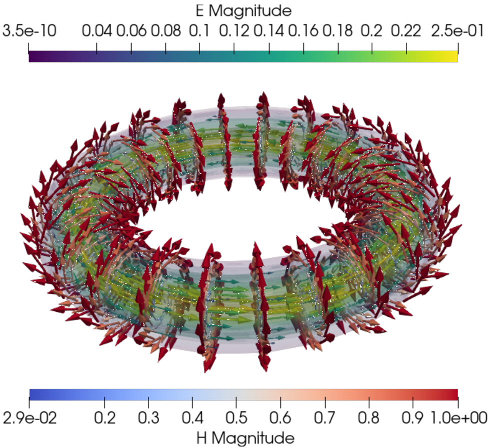
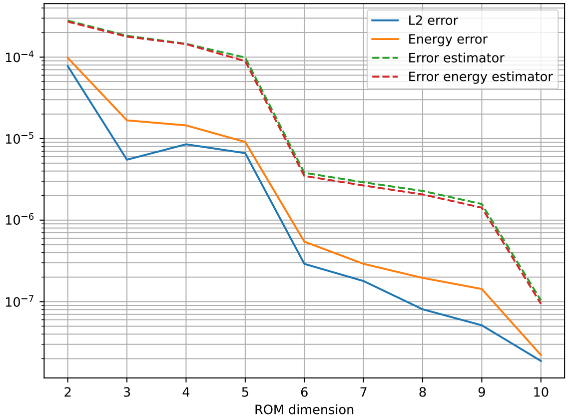


Figure: Simulation for $\sigma = \mu = 1$

Curl-Curl (Maxwell): Error Decay



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Curl-Curl (Maxwell): Speed-up

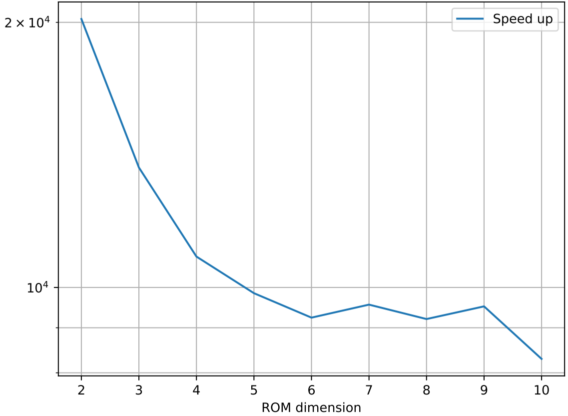

Figure: Speed up factor for different number of reduced basis N_{RB}

Curl-Curl (Maxwell): Exploring parameter space

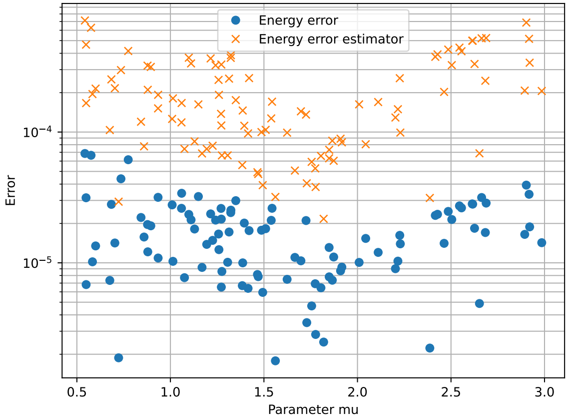

Figure: Error in energy norm and its estimator on a test set with respect to μ for $N_{RB} = 4$

Curl-Curl (Maxwell): Simulations

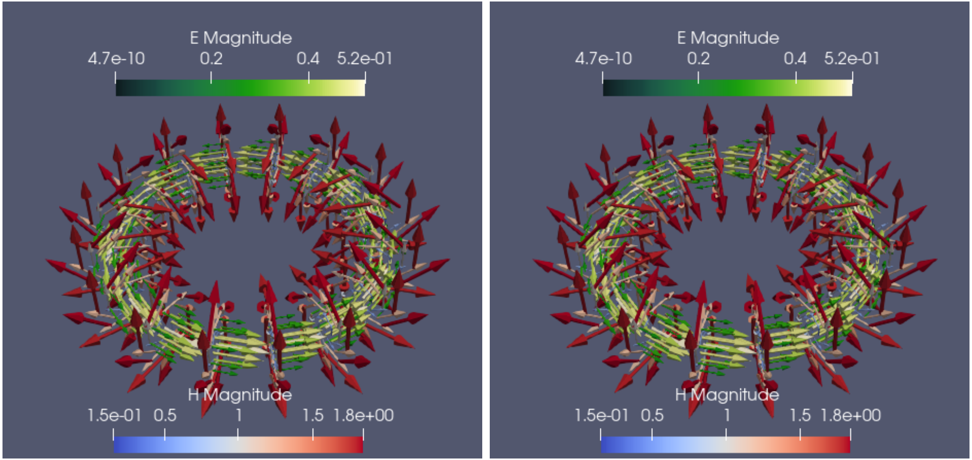

Figure: Simulation for $\sigma = 0.994$, $\mu = 1.94$, FOM left ROM ($N_{RB} = 4$) right

Curl-Curl (Maxwell): Simulations

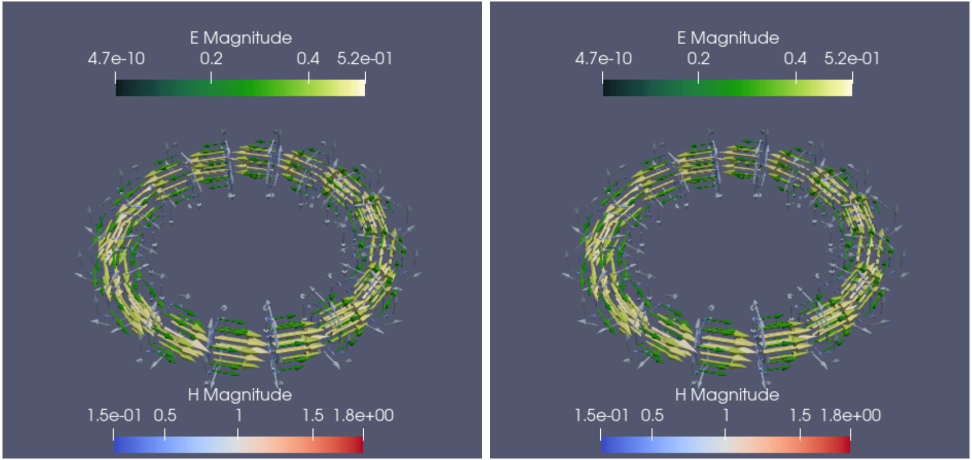

Figure: Simulation for $\sigma = 0.529$, $\mu = 0.745$, FOM left ROM ($N_{RB} = 4$) right

Table of contents

Summary and perspectives

Summary

- Friedrichs' Systems
- Linear **hyperbolic** problems
- Generalize many models
- Extra variables
- Discretization **DG**
- ROM with Galerkin projection
- **Error estimator(s)**
- Great **speed-up**

Perspectives

- Error estimator in energy norm (very similar)
- **Ultraweak** formulation to obtain a better error estimator (=residual)
- Linearized **nonlinear** systems (Euler)
- Slow Kolmogorov N_{RB}-width decay
- **Time**-dependent systems

Literature

- K. O. Friedrichs. Symmetric positive linear differential equations. Communications on Pure and Applied Mathematics, 11(3):333–418, 1958.
- A. Ern and J.-L. Guermond. Discontinuous Galerkin methods for Friedrichs' systems. I. General theory. SIAM journal on numerical analysis, 44(2):753–778, 2006.
- D. A. Di Pietro and A. Ern. Mathematical aspects of discontinuous Galerkin methods, volume 69. Springer Science & Business Media, 2011.
- T. Sonar and E. Süli. A dual graph-norm refinement indicator for finite volume approximations of the Euler equations. Numerische Mathematik, 78(4):619–658, 1998.
- E. Beurer, M. Feuerle, N. Reich, and K. Urban. An ultraweak variational method for parameterized linear differential-algebraic equations. arXiv preprint arXiv:2202.12834, 2022.
- P. Lesaint. Finite element methods for symmetric hyperbolic equations. Numerische Mathematik, 21(3):244–255, 1973.
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THANK YOU!

DG discretization

- $V_h := [\mathbb{P}^p_d(\mathcal{T}_h)]^m$
- Hence, $\mathcal{D}|_T := \sum_{k=1}^d n_{T,k}\mathcal{A}^k|_T$ on ∂T
- $\mathcal{D}_F[\![z^{ex}]\!] = 0$
- $$a_h^{cf}(z,y_h) := \sum_{T\in\mathcal{T}_h}(Az,y_h)_{L^2(T)} + \frac{1}{2}\sum_{F\in\mathcal{F}^b}((\mathcal{M}-\mathcal{D})z,y_h)_{L^2(F)} - \sum_{F\in\mathcal{F}_h^i}(\mathcal{D}_F[\![z]\!],\{\{y_h\}\})_{L^2(F)}$$
- Consistent and coercive:
$$a_h^{cf}(y_h,y_h) \geq \mu_0||y_h||_L^2 + \frac{1}{2}|y_h|_M^2$$
- Well-posed discrete problem (not optimal convergence rate)
- Stabilization $a_h(z,y_h) := a_h^{cf}(z,y_h) + s_h(z,y_h)$
$$s_h(z,y_h) := \sum_{F\in\mathcal{F}_h^b}(S_F^b z,y_h)_{L^2(F)} + \sum_{F\in\mathcal{F}_h^i}(S_h^i[\![z]\!],[\![y_h]\!])_{L^2(F)}$$

- $\mathcal{A}^k|_T \in [\mathcal{C}^{0,\frac{1}{2}}(\overline{T})]^{m,m}$
- Define a unique $\mathcal{D}_F$ for every face F

DG discretization

- a_h
  - Consistent, coercive
  - Inf-sup stability

$$|||z_h||| \lesssim \sup_{y_h \neq 0} \frac{a_h(z_h, y_h)}{|||y_h|||}$$

$$|||y|||^2 := ||y||^2_{L^2} + |y|^2_M + |y|^2_S + \sum_{T \in \mathcal{T}_h} h_T \left\| \sum_{k=1}^{d} A^k \partial_k y \right\|_{L^2(T)}$$

  - Boundness $a_h(w, y_h) \lesssim |||w|||_* |||y_h|||$ with

$$|||y|||^2_* = |||y|||^2 + \sum_{T \in \mathcal{T}_h} (h_T^{-1} ||y||^2_{L^2(T)} + ||y||^2_{L^2(\partial T)})$$

- **Error estimate**

$$|||z - z_h||| \lesssim \inf_{y_h \in V_h} |||z - y_h|||_*$$

$$|||z - z_h||| \lesssim C_z h^{k+1/2}$$

DG for Friedrichs' Systems

DG discretization

- Stable
- High Order Accurate
- Linear discretization of the linear operators
- Easy parallel implementation of the system assembly
- Affine decomposition easy to exploit
- Implemented in `deal.II`[3]

[3] `dealii.org`

ROM methods

POD

- Ξ training set of μ
- Compute $z_h(\mu) \quad \forall \mu \in \Xi$
- Compute the SVD of Z the snapshot matrix
- Retain the largest singular values $\hat{\Sigma}$ and vectors $\hat{W}$ (using a tolerance or a fixed number)
- Set $V_{RB} := \hat{W}$

Greedy algorithm

- Ξ training set of μ
- Start with random $\mu^0 \in \Xi$ and $V_{RB} := z_h(\mu^0)$
- Compute the maximum of the error estimator $\Delta_{N_{RB}}^{max} = \max_{\mu \in \Xi} \Delta_{2,RB}(\mu)$
- While $\Delta_{N_{RB}}^{max} > \mathrm{tol}$
  - $\mu^* = \arg\max_{\mu \in \Xi} \Delta_{2,RB}(\mu)$
  - $V_{RB} = V_{RB} \oplus z_h(\mu^*)$

Error bound

- $e_h(\mu) := z_h(\mu) - z_{RB}(\mu)$
- $\langle u_h, v_h \rangle_2 := \sum_{T\in\mathcal{T}_h} \int_T u_h v_h dx$
- $r^\mu_{RB}(y_h) := (f, y_h) - a_h(z_{RB}(\mu), y_h) = (f, y_h) - a_h(z_{RB}(\mu), y_h) - (f, y_h) + a_h(z_h(\mu), y_h) = a_h(e_h(\mu), y_h)$
- Let $\hat{r}^\mu_{RB}$ be the Riesz representing vector in $\langle \cdot, \cdot \rangle_2$ of $r^\mu_{RB}(\cdot)$, i.e.,

$$r^\mu_{RB}(y_h) = \langle \hat{r}^\mu_{RB}, y_h \rangle_2$$

- Recall that

$$a_h(e_h, e_h) \geq \sum_{T\in\mathcal{T}_h} (Ae_h, e_h)_{L^2(T)} \geq \mu_0 ||e_h||_2^2$$

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$$||e_h(\mu)||_2^2 \leq \frac{1}{\mu_0} a_h(e_h(\mu), e_h(\mu); \mu) = \frac{1}{\mu_0} r^\mu_{RB}(e_h(\mu)) = \frac{1}{\mu_0} \langle \hat{r}^\mu_{RB}, e_h(\mu) \rangle_2 \leq \frac{1}{\mu_0} ||\hat{r}^\mu_{RB}||_2 ||e_h(\mu)||_2$$

$$||e_h(\mu)||_2 \leq \frac{||\hat{r}^\mu_{RB}||_2}{\mu_0} =: \Delta_{2,RB}(\mu)$$

A posteriori error estimator

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A posteriori error estimator

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A posteriori error estimator

Effectiveness error estimator

Boundness of a_h depends on norm $|||\cdot|||_*$, hence, very bad estimates (we are working on that)

Cheap computation of the error estimator

$$\hat{r}^{\mu}_{RB} := F - Az_{RB} = \sum_{\ell=1}^{N^f_{affine}} \theta^f_\ell(\mu) F_\ell - \sum_{k=1}^{N_{affine}} \theta_k(\mu) A_k \left(\sum_{n=1}^{N_{RB}} z^n_{RB} \psi^{RB}_n \right)$$

$$||\hat{r}^{\mu}_{RB}||^2_2$$

$$:= \left\langle \sum_{\ell=1}^{N^f_{affine}} \theta^f_\ell(\mu) F_\ell - \sum_{k=1}^{N_{affine}} \theta_k(\mu) A_k \left(\sum_{n=1}^{N_{RB}} z^n_{RB} \psi^{RB}_n \right), \sum_{r=1}^{N^f_{affine}} \theta^f_r(\mu) F_r - \sum_{s=1}^{N_{affine}} \theta_s(\mu) A_s \left(\sum_{m=1}^{N_{RB}} z^m_{RB} \psi^{RB}_m \right) \right\rangle_2$$

$$= \underbrace{\sum_{k,s=1}^{N_{affine}} \sum_{n,m=1}^{N_{RB}} \theta_k(\mu) z^n_{RB} \theta_s(\mu) z^m_{RB}}_{\text{online}} \underbrace{\left\langle A_k \psi^{RB}_n, A_s \psi^{RB}_m \right\rangle_2}_{\text{offline}} + \dots$$