Global Flux WENO FV and other structure preserving schemes for water equations

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Motivation[1]

Goal

$$\partial_t \mathbf{u} + \partial_x \mathcal{F}(\mathbf{u}) = \mathcal{S}(x,\mathbf{u}) \quad \xRightarrow{\partial_t \mathbf{u}\to 0} \quad \partial_x \mathcal{F}(\mathbf{u}) = \mathcal{S}(x,\mathbf{u}) \text{ (Non-trivial steady states)}$$

- **Shallow water equations with topography, friction, . . .**
- Euler equations with gravity
- Shallow Water/Euler in pseudo-1D form (section variation)

[1] with Mirco Ciallella and Mario Ricchiuto

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- **Shallow water equations with topography, friction, . . .**
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State-of-the-art

- Reference solution (Berberich *et al.*. Comp. Flui. 2021.)
- Hydrostatic reconstruction (Castro *et al.*. Math. Mod. Meth. Appl. Sci. 2007.)
- Modified Riemann solvers (Michel-Dansac *et al.*. Jour. Comp. Phys. 2017.)

Our contribution

“Special quadrature” of the source terms

- **Arbitrary high** order framework
- Schemes **agnostic** of general moving equilibria
- Preservation of both continuous and **discontinuous equilibria**
- Easy to generalize to other equilibria

[1] with Mirco Ciallella and Mario Ricchiuto

Model

Shallow Water Equations (SWE)

$$\partial_t \mathbf{u} + \partial_x \mathcal{F}(\mathbf{u}) = \mathcal{S}(\mathbf{u}, x), \qquad \text{on} \qquad \boldsymbol{\Omega}_T = \boldsymbol{\Omega} \times [0, T] \subset \mathbb{R} \times \mathbb{R}^+.$$

$$\mathbf{u} = \begin{bmatrix} h \\ q \end{bmatrix}, \quad \mathcal{F}(\mathbf{u}) = \begin{bmatrix} q \\ \frac{q^2}{h} + g\frac{h^2}{2} \end{bmatrix},$$

$$\mathcal{S}(\mathbf{u}, x) = \begin{bmatrix} 0 \\ S(\mathbf{u}, x) \end{bmatrix} = -gh \begin{bmatrix} 0 \\ \frac{\partial b(x)}{\partial x} \end{bmatrix} - gq \begin{bmatrix} 0 \\ \frac{n^2}{h^{7/3}}|q| \end{bmatrix}$$

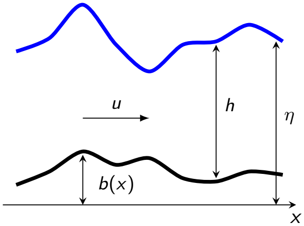
where

$h =$ *water height*, $q =$ *discharge* $(= hu)$,

$u =$ *velocity*, $g =$ *gravity*,

$b =$ *bathymetry*, $n =$ *Manning friction*,

$\eta =$ *free surface level*



Steady state equilibria

- Classical discretization results in *numerical storms*.
  $\Longrightarrow$ **well-balanced (WB) schemes**

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- Non-trivial **moving water equilibria**:
- Strong Form (smooth solutions)

$$\begin{cases} q(x,t) = h(x,t)u(x,t) \equiv q_0 \\ \mathcal{E}(\mathbf{u},x) = \frac{1}{2}u^2 + g(h+b) \equiv \mathcal{E}_0 \end{cases}$$

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- ∘ Weak Form

$$\begin{cases} q(x,t) = h(x,t)u(x,t) \equiv q_0 \\ K(\mathbf{u},x) = \frac{q^2}{h} + g\frac{h^2}{2} + \int_{x_0}^{x} g\left[h\partial_\xi b + \frac{n^2|q|q}{h^{7/3}}\right] \mathrm{d}\xi \equiv K_0. \end{cases}$$

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Global Flux idea[2,3]

Shallow Water Equations (SWE)

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[2] Gascon, Corberan. Jour. Comp. Phys. 2001.
[3] Cheng *et al.*. Jour. Sci. Comp. 2019.

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Global Flux SWE

$$\partial_t \mathbf{u} + \partial_x \mathcal{G}(\mathbf{u}, x) = 0, \qquad \mathcal{G}(\mathbf{u}, x) = \mathcal{F}(\mathbf{u}) + \begin{bmatrix} 0 \\ \mathcal{R}(\mathbf{u}, x) \end{bmatrix} = \begin{bmatrix} q \\ K \end{bmatrix} = \begin{bmatrix} q \\ \frac{q^2}{h} + g\frac{h^2}{2} + \mathcal{R} \end{bmatrix}$$

$$\mathcal{R}(\mathbf{u}, x) := -\int^x S(\mathbf{u}, \xi)\, \mathrm{d}\xi = g \int^x \left[h(\xi, t)\frac{\partial b(\xi)}{\partial \xi} + \frac{n^2}{h^{7/3}(\xi, t)}|q(\xi, t)|q(\xi, t) \right] \mathrm{d}\xi$$

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Global Flux Finite Volume method

$$\partial_t \mathbf{u} + \partial_x \mathcal{G}(\mathbf{u}, x) = 0$$

Finite Volume

- $\boldsymbol{\Omega}$ is discretized into N_x control volumes $\Omega_i = [x_{i-1/2}, x_{i+1/2}]$ of size Δx centered at $x_i = i\Delta x$ with $i = i_\ell, \dots, i_r$.
- **Cell average** at time t:

$$\bar{\mathbf{U}}_i(t) := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}(x,t)\,\mathrm{d}x.$$

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$$\bar{\mathbf{U}}_i(t) := \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{u}(x,t)\,\mathrm{d}x.$$
- Semi-discrete finite volume:
$$\frac{\mathrm{d}\bar{\mathbf{U}}_i}{\mathrm{d}t} + \frac{1}{\Delta x}(\mathbf{H}_{i+1/2} - \mathbf{H}_{i-1/2}) = 0$$
where $\mathbf{H}_{i\pm1/2}$ is a **numerical flux** consistent with the flux $\mathcal{G}$.

Global Flux Finite Volume method: numerical flux

Numerical Flux

- $\mathbf{H}_{i+1/2}$ **upwind numerical flux** defined only on the **global flux**:

$$\mathbf{H}_{i+1/2} = \mathcal{L}^{-1}\Lambda^{+}\mathcal{L}\,\mathcal{G}^{L}_{i+1/2} + \mathcal{L}^{-1}\Lambda^{-}\mathcal{L}\,\mathcal{G}^{R}_{i+1/2}.$$

- $\mathcal{G}^{L,R}_{i+1/2} =$ the **discontinuous** reconstructed values of $\mathcal{G}$
- $\mathcal{L} =$ left eigenvectors computed with the Roe's state[4]
- $\Lambda^{\pm} =$ upwinding weights such that

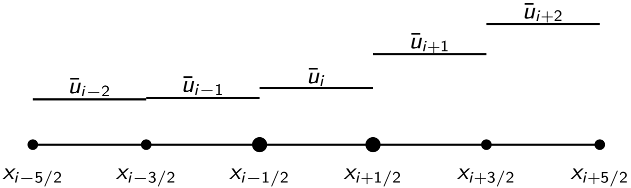
$$\Lambda^{+}_i = \begin{cases} 1, & \text{if } \lambda_i > 0, \\ 0, & \text{if } \lambda_i < 0, \end{cases} \qquad \Lambda^{-}_i = \begin{cases} 1, & \text{if } \lambda_i < 0, \\ 0, & \text{if } \lambda_i > 0. \end{cases}$$

- Rusanov not directly possible (it depends on variable $\mathbf{u}$ instead of $\mathcal{G}$)

[4] Roe. Approximate Riemann solvers, parameter vectors, and difference schemes. Jour. Comp. Phys. 1981.

Global Flux Finite Volume method: high order WENO reconstruction

Reconstructed values in ξ are computed using WENO [5]



- p^{HO} high order polynomial
- p_j low order polynomials
- β_j smoothness indicators

- Optimal weights d_j^ξ:
  $\sum_j d_j^\xi p_j(\xi) = p^{HO}(\xi)$
- Nonlinear weights
  $\omega_j^\xi = \frac{d_j^\xi}{(\beta_j+\varepsilon)^2}$

[5] Jiang, and Shu. Efficient implementation of weighted ENO schemes. Jour. Comp. Phys. 1996.

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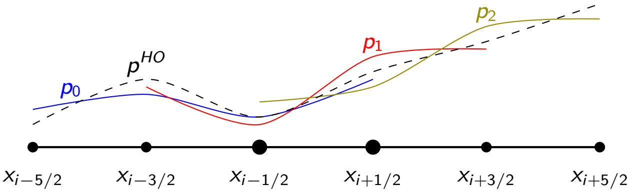

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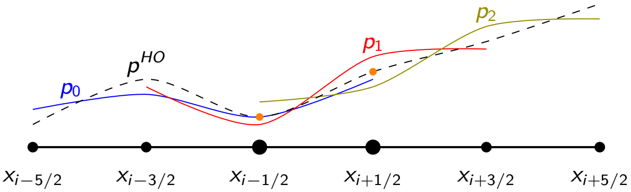

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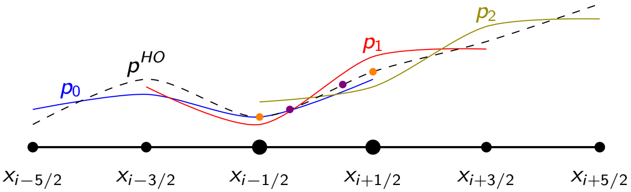

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Global Flux Finite Volume method: Global Flux Reconstruction

Global Flux Reconstruction

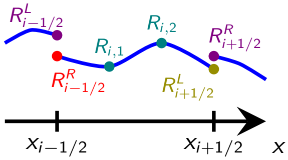
- To reconstruct $\mathcal{G}$ at the interfaces, we need the cell averages $\bar{\mathcal{G}}_i$

$$\bar{\mathcal{G}}_i(\mathbf{u},x) = \bar{\mathcal{F}}_i(\mathbf{u}) + \begin{bmatrix} 0 \\ \bar{\mathcal{R}}_i \end{bmatrix} \quad \text{with} \quad \begin{cases} \bar{\mathcal{F}}_i(\mathbf{u}) & \approx \sum_\omega w_\omega \mathcal{F}(\mathbf{u}(x_{i,\omega})) \\ \bar{\mathcal{R}}_i & \approx \sum_\omega w_\omega \mathcal{R}_{i,\omega} \end{cases}.$$

- Compute $\mathcal{R}$ in the quadrature points, using an iterative procedure

$$\mathcal{R}_{i,\omega} = \mathcal{R}^R_{i-1/2} + \int_{x^R_{i-1/2}}^{x_{i,\omega}} \tilde{S}(\mathbf{u},x)\,\mathrm{d}x$$

$$= \mathcal{R}^R_{i-1/2} + \sum_\theta \underbrace{\int_{x^R_{i-1/2}}^{x_{i,\omega}} \ell_\theta(x)\,\mathrm{d}x}_{r^\omega_\theta}\; S(\mathbf{u}_{i,\theta}, x_{i,\theta}), \quad i > i_\ell.$$



Global Flux Finite Volume method: Global Flux Reconstruction

Global Flux Reconstruction

- Iterative procedure to obtain $\mathcal{R}$ on both sides of each interface

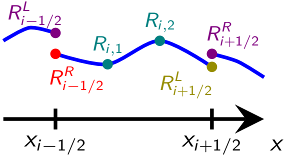
$$\mathcal{R}^L_{i+1/2} = \mathcal{R}^R_{i-1/2} + \int_{x^R_{i-1/2}}^{x^L_{i+1/2}} S(\mathbf{u}, x)\,\mathrm{d}x$$
$$= \mathcal{R}^R_{i-1/2} + \Delta x \bar{S}_i, \quad i > i_\ell.$$

$\bar{S}_i$ being the cell average of S computed as

$$\bar{S}_i := \frac{1}{\Delta x}\int_{x^R_{i-1/2}}^{x^L_{i+1/2}} S(\mathbf{u}, \xi)\,\mathrm{d}\xi \approx \sum_\omega w_\omega S(\mathbf{u}_{i,\omega}, x_{i,\omega}).$$

- Recursive definition of the $\mathcal{R}^R_{i+1/2}$ interface value (discontinuous bathymetry definition)

$$\mathcal{R}^R_{i+1/2} = \mathcal{R}^L_{i+1/2} + [\![\mathcal{R}_{i+1/2}]\!].$$



What is still missing

Formulae we want

- $\bar{\mathcal{R}}_i \approx \sum_\omega w_\omega \mathcal{R}_{i,\omega}$;
- $\mathcal{R}_{i,\omega} = \mathcal{R}^R_{i-1/2} + \sum_\theta r^\omega_\theta \, S(\mathbf{u}_{i,\theta}, x_{i,\theta}), \quad i > i_\ell$;
- $\mathcal{R}^L_{i+1/2} = \mathcal{R}^R_{i-1/2} + \Delta x \bar{S}_i, \quad i > i_\ell$;
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What we have

- Global Flux discretization
- In any steady equilibria we preserve $q \equiv q_0$ and $K \equiv K_0$

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Formulae we want

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What we have

- Global Flux discretization
- In any steady equilibria we preserve $q \equiv q_0$ and $K \equiv K_0$

What we do not have

- Preservation of $\eta \equiv \eta_0$ in lake at rest
- Discretization of $S = -gh\partial_x b - g\frac{q|q|n^2}{h^{7/3}}$ in quadrature points (bathymetry)
- Definition of $[\![\mathcal{R}_{i+1/2}]\!]$

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Formulae we want

- $\bar{\mathcal{R}}_i \approx \sum_\omega w_\omega \mathcal{R}_{i,\omega}$;
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What we want to achieve

- Well balancing also for **lake at rest**
- High order discretization
- Keeping the global flux formulation
- Definition of $\mathcal{R}_{i,\omega}$, $\mathcal{R}^L_{i+1/2}$, $[\![\mathcal{R}_{i+1/2}]\!]$

What we have

- Global Flux discretization
- In any steady equilibria we preserve $q \equiv q_0$ and $K \equiv K_0$

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- Preservation of $\eta \equiv \eta_0$ in lake at rest
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Well-balanced for the lake at rest

Well balanced for lake at rest ($\eta \equiv \eta_0$)

- WENO reconstruction of h, η and b with the weights from η.
- We highlight η in the source term, given that $h(x) = \eta(x) - b(x)$, as

$$S(\mathbf{u}, x) = gh\partial_x b = g(\eta - b)\partial_x b = g\eta\partial_x b - g\partial_x \left(\frac{b^2}{2}\right).$$

How to proceed

- Substitute the new form of S into the definitions of $\mathcal{R}_{i,\omega}, \mathcal{R}_i^L$
- Check what we get when $\eta \equiv \eta_0$
- Defining remaining terms so that we are **WB** for lake at rest

Well-balanced for the lake at rest

Substitution ($\eta \equiv \eta_0$)

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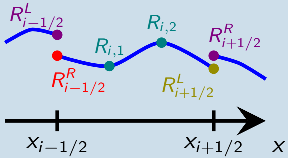


- $\mathcal{R}^L_{i+1/2}$ reads

$$\mathcal{R}^L_{i+1/2} = \mathcal{R}^R_{i-1/2} - \int_{x^R_{i-1/2}}^{x^L_{i+1/2}} S(\mathbf{u}(x),x)\,\mathrm{d}x$$

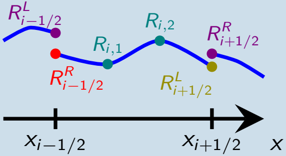
$$= \mathcal{R}^R_{i-1/2} + g\int_{x^R_{i-1/2}}^{x^L_{i+1/2}} \eta(x)\partial_x b(x)\,\mathrm{d}x - g\left(\frac{(b^L_{i+1/2})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right)$$

$$(\text{if } \eta \equiv \eta_0) = \mathcal{R}^R_{i-1/2} + g\eta_0\left(b^L_{i+1/2} - b^R_{i-1/2}\right) - g\left(\frac{(b^L_{i+1/2})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right).$$

Well-balanced for the lake at rest

Substitution ($\eta \equiv \eta_0$)

$$S(\mathbf{u},x) = gh\partial_x b = g(\eta - b)\partial_x b = g\eta\partial_x b - g\partial_x\left(\frac{b^2}{2}\right).$$



- $\mathcal{R}_{i,\omega}$ reads

$$\begin{aligned}
\mathcal{R}_{i,\omega} &= \mathcal{R}^R_{i-1/2} - \int_{x^R_{i-1/2}}^{x_{i,\omega}} S(\mathbf{u}(x),x)\,\mathrm{d}x \\
&= \mathcal{R}^R_{i-1/2} + g\int_{x^R_{i-1/2}}^{x_{i,\omega}} \eta(x)\partial_x b(x)\,\mathrm{d}x - g\left(\frac{(b_{i,\omega})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right) \\
\text{(if } \eta \equiv \eta_0\text{)} &= \mathcal{R}^R_{i-1/2} + g\eta_0\left(b_{i,\omega} - b^R_{i-1/2}\right) - g\left(\frac{(b_{i,\omega})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right).
\end{aligned}$$

Well-balanced for the lake at rest

Continue computation with $\eta \equiv \eta_0,\ q \equiv 0$

$$K_{i,\omega} = \mathcal{F}_{i,\omega} + \mathcal{R}_{i,\omega} = g\frac{(\eta_0 - b_{i,\omega})^2}{2} + \mathcal{R}^R_{i-1/2} + g\eta_0\left(b_{i,\omega} - b^R_{i-1/2}\right) - g\left(\frac{(b_{i,\omega})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right) =$$

$$= \mathcal{R}^R_{i-1/2} + g\frac{\eta_0^2}{2} - g\eta_0 b^R_{i-1/2} + g\frac{(b_{i-1/2})^2}{2}.$$

Independent of ω, hence, $\bar{K}_i = \mathcal{R}^R_{i-1/2} + g\frac{\eta_0^2}{2} - g\eta_0 b^R_{i-1/2} + g\frac{(b_{i-1/2})^2}{2}$.
We now would like $\bar{K}_i \equiv K_0$ for all i.

$$\begin{aligned}\bar{K}_{i+1} - \bar{K}_i \ &= \mathcal{R}^R_{i+1/2} - \mathcal{R}^R_{i-1/2} - g\eta_0 b^R_{i+1/2} + g\frac{(b^R_{i+1/2})^2}{2} + g\eta_0 b^R_{i-1/2} - g\frac{(b^R_{i-1/2})^2}{2}\\ &= \mathcal{R}^L_{i+1/2} - \mathcal{R}^R_{i-1/2} + [\![\mathcal{R}_{i+1/2}]\!] - g\eta_0 b^R_{i+1/2} + g\frac{(b^R_{i+1/2})^2}{2} + g\eta_0 b^R_{i-1/2} - g\frac{(b^R_{i-1/2})^2}{2} = \dots\end{aligned}$$

Well-balanced for the lake at rest

Continue computation with $\eta \equiv \eta_0$, $q \equiv 0$

$$\begin{aligned}
\bar{K}_{i+1} - \bar{K}_i =& \mathcal{R}^L_{i+1/2} - \mathcal{R}^R_{i-1/2} + [\![\mathcal{R}_{i+1/2}]\!] - g\eta_0 b^R_{i+1/2} + g\frac{(b^R_{i+1/2})^2}{2} + g\eta_0 b^R_{i-1/2} - g\frac{(b^R_{i-1/2})^2}{2} \\
=& \underbrace{g\eta_0 \left(b^L_{i+1/2} - b^R_{i-1/2}\right) - g\left(\frac{(b^L_{i+1/2})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right)}_{\mathcal{R}^L_{i+1/2} - \mathcal{R}^R_{i-1/2}} + [\![\mathcal{R}_{i+1/2}]\!] \\
& - g\eta_0 b^R_{i+1/2} + g\frac{(b^R_{i+1/2})^2}{2} + g\eta_0 b^R_{i-1/2} - g\frac{(b^R_{i-1/2})^2}{2} = \\
=& g\eta_0 \left(b^L_{i+1/2} - b^R_{i-1/2}\right) - g\left(\frac{(b^L_{i+1/2})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right) + [\![\mathcal{R}_{i+1/2}]\!] = 0,
\end{aligned}$$

Hence, we define

$$[\![\mathcal{R}_{i+1/2}]\!] := g\frac{\eta^R_{i+1/2} + \eta^L_{i+1/2}}{2}\left(b^R_{i+1/2} - b^L_{i+1/2}\right) - g\left(\frac{(b^R_{i+1/2})^2}{2} - \frac{(b^L_{i+1/2})^2}{2}\right).$$

Summary of the method

Global Flux WENO FV method

- $\mathcal{R}_{i_\ell-1/2} := 0$, then for every cell i
  - Reconstruct h, η and b in each quadrature point $\Rightarrow \tilde{h}_{i,\theta}$, $\tilde{\eta}_{i,\theta}$ and $\tilde{b}_{i,\theta}$
  - Reconstruct q in the quadrature points obtaining $\tilde{q}_{i,\theta}$
  - $\mathcal{R}_{i,\omega} = \mathcal{R}^R_{i-1/2} + g \sum_\theta \int_{x^R_{i-1/2}}^{x_{i,\omega}} \ell_\theta(x)\,\mathrm{d}x \left(\tilde{\eta}_{i,\theta} \sum_s \ell'_s(x_{i,\theta})\tilde{b}_{i,s} + g\frac{\tilde{q}_{i,\theta}|\tilde{q}_{i,\theta}|n^2}{\tilde{h}^{7/3}_{i,\theta}}\right) - g\left[\frac{(b_{i,\theta})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right]$
  - $\mathcal{R}^L_{i+1/2} = \mathcal{R}^R_{i-1/2} + g \sum_\theta \int_{x^R_{i-1/2}}^{x_{i+1/2}} \ell_\theta(x)\mathrm{d}x \left(\tilde{\eta}_{i,\theta} \sum_s \ell'_s(x_{i,\theta})\tilde{b}_{i,s} + g\frac{\tilde{q}_{i,\theta}|\tilde{q}_{i,\theta}|n^2}{\tilde{h}^{7/3}_{i,\theta}}\right) - g\left[\frac{(b^L_{i+1/2})^2}{2} - \frac{(b^R_{i-1/2})^2}{2}\right].$
  - $[\![\mathcal{R}_{i+1/2}]\!] := g\frac{\eta^R_{i+1/2}+\eta^L_{i+1/2}}{2}\left(b^R_{i+1/2} - b^L_{i+1/2}\right) - g\left(\frac{(b^R_{i+1/2})^2}{2} - \frac{(b^L_{i+1/2})^2}{2}\right).$
  - $\mathcal{R}^R_{i+1/2} := \mathcal{R}^L_{i+1/2} + [\![\mathcal{R}_{i+1/2}]\!]$

Properties

- Preserves moving equilibria ($q \equiv q_0$ and $K \equiv K_0$) • Preserves lake at rest equilibria (also $\eta \equiv \eta_0$)

High order time discretization

Deferred Correction Method

- ☺ Arbitrarily high order
- ☺ Explicit method (there exists also implicit)
- Based on two operators
  - High order implicit operator, we do not invert
  - Explicit low order operator, we solve
- Based on iterations (as many as the order)
- Similar to ADER prediction step
- Can be written as a RK
- ☹ Many RK stages (see Lorenzo's talk in 1 hour)
- We test 3^{rd} order 5 stages and 5^{th} order 13 stages

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Validation: Lake at rest

Domain and Bathymetry

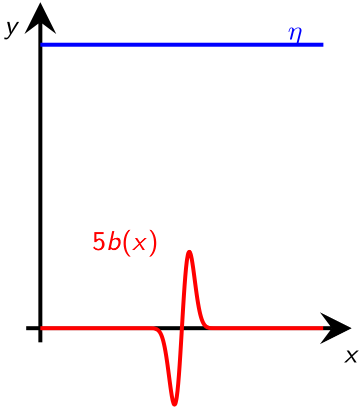
$$\Omega = [0, 25],$$
$$b(x) = 0.05 \sin(x - 12.5) \exp\left(1 - (x - 12.5)^2\right),$$
$$g = 9.812.$$

$b(x)$ is chosen $\mathcal{C}^\infty$ and such that it has values smaller than machine precision at the boundaries.

Lake at rest test

$$h(x,0) = 1 - b(x), \qquad q(x,0) \equiv 0$$

BC: subcritical inflow/outflow



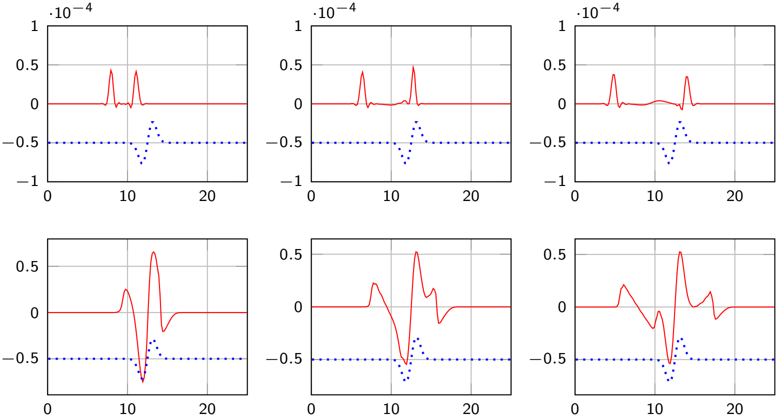
Validation: Lake at rest

Table: Lake at rest: errors and estimated order of accuracy (EOA) with WB and non-WB schemes and GF-WENO3 and GF-WENO5.

	Non-WB				WB			
	h		q		h		q	
N_e	L_2 error	EOA	L_2 error	EOA	L_2 error	EOA	L_2 error	EOA
	GF-WENO3				GF-WENO3			
25	1.0384E-4	–	4.7943E-5	–	9.8858E-14	–	1.2228E-15	–
50	1.5496E-5	2.67	9.2488E-6	2.31	9.8667E-14	–	1.4249E-15	–
100	1.2117E-6	3.62	3.6777E-7	4.59	9.8276E-14	–	1.6041E-15	–
150	2.6776E-7	3.69	1.5898E-7	2.05	1.9644E-13	–	3.3908E-15	–
200	9.6323E-8	3.53	7.6469E-8	2.53	1.9619E-13	–	3.6713E-15	–
400	8.2671E-9	3.53	6.0441E-9	3.65	2.9360E-13	–	6.1689E-15	–
800	6.8811E-10	3.58	4.7122E-10	3.67	5.8655E-13	–	1.3035E-14	–
	GF-WENO5				GF-WENO5			
25	5.1800E-5	–	6.1657E-5	–	9.8947E-14	–	1.3247E-15	–
50	4.4066E-6	3.45	1.5244E-6	5.18	9.8661E-14	–	1.4060E-15	–
100	6.7998E-7	2.66	3.5908E-7	2.06	9.8289E-14	–	1.5992E-15	–
150	1.5437E-7	3.63	8.8535E-8	3.42	1.9639E-13	–	3.4157E-15	–
200	4.1973E-8	4.50	2.3725E-8	4.55	1.9611E-13	–	3.7034E-15	–
400	1.3952E-9	4.89	7.5991E-10	4.95	2.9357E-13	–	6.2007E-15	–
800	4.3120E-11	5.01	2.2633E-11	5.06	5.8648E-13	–	1.3039E-14	–

Validation: Perturbed lake at rest

GF-WENO5 (top) and WENO5 (bottom): $h - h_{eq}$ (red) and rescaled b (blue)



(d) $t = 0.5$ (e) $t = 1$ (f) $t = 1.5$

Validation: Steady states without friction ($n = 0$)

Supercritical flow test

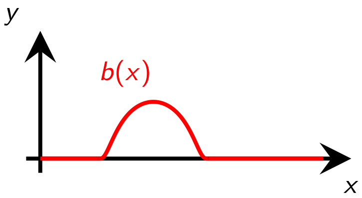
$$h(x,0) = 2 - b(x), \qquad q(x,0) \equiv 0,$$
$$h(0,t) = 2, \qquad q(0,t) = 24,$$

Subcritical flow test

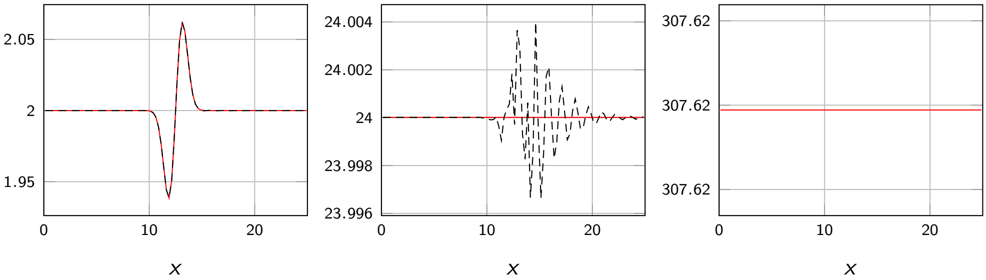
$$h(x,0) = 2 - b(x), \qquad q(x,0) \equiv 0,$$
$$q(0,t) = 4.42, \qquad h(25,t) = 2,$$

Transcritical flow test

$$b(x) = \begin{cases} 0.2 \exp\left(1 - \frac{1}{1-\left(\frac{|x-10|}{5}\right)^2}\right), & \text{if } |x-10| < 5, \\ 0, & \text{else,} \end{cases}$$
$$h(x,0) = 0.33 - b(x), \qquad q(x,0) \equiv 0,$$
$$q(0,t) = 0.18, \qquad h(25,t) = 0.33.$$



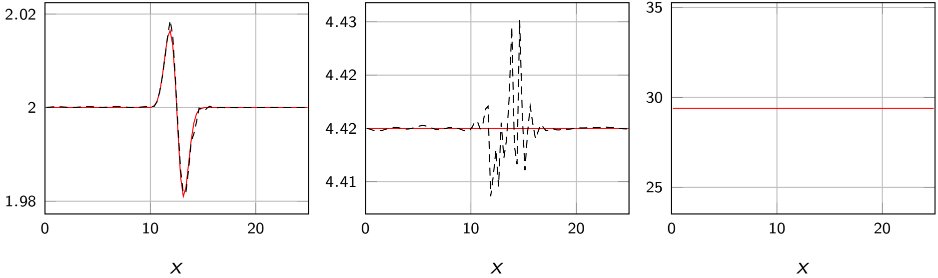
Validation: Smooth steady states without friction ($n = 0$)



(g) η (h) q (i) K

Figure: Supercritical flow: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes with $N_e = 100$.

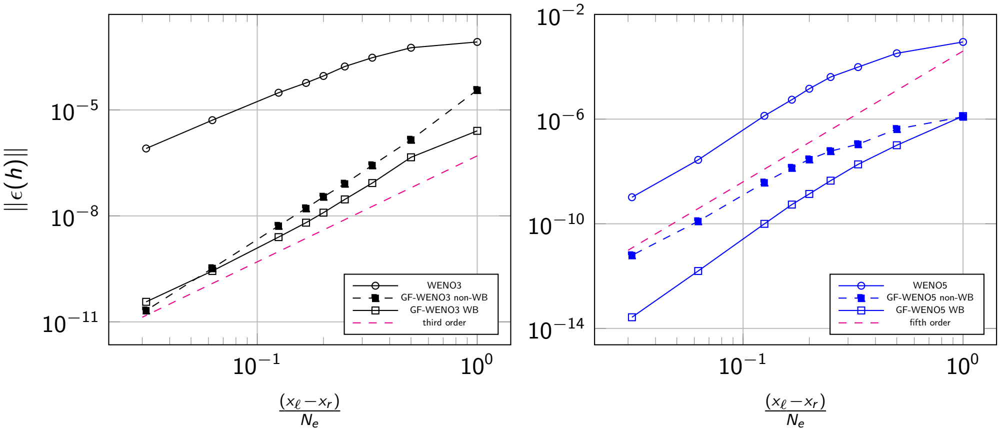
Validation: Smooth steady states without friction ($n = 0$)



(a) η (b) q (c) K

Figure: Subcritical flow: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes with $N_e = 100$.

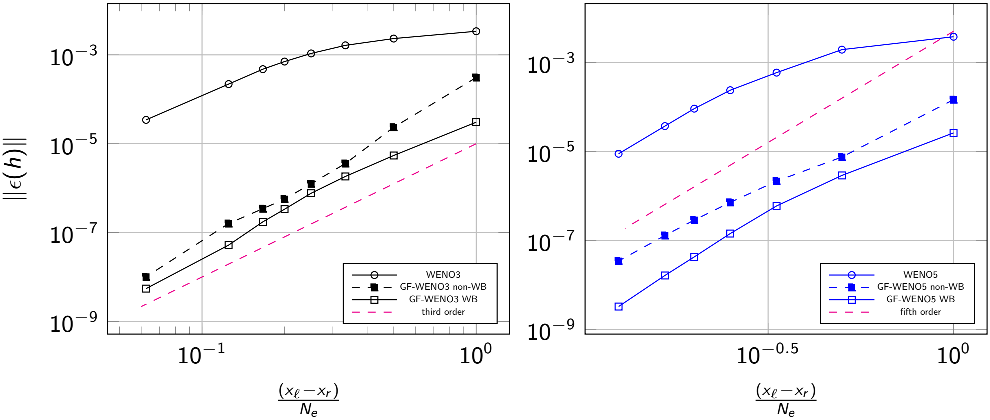
Validation: Smooth steady states without friction ($n = 0$)



(a) WENO3 (b) WENO5

Figure: Supercritical flow: convergence tests with WENO3 and WENO5.

Validation: Smooth steady states without friction ($n = 0$)

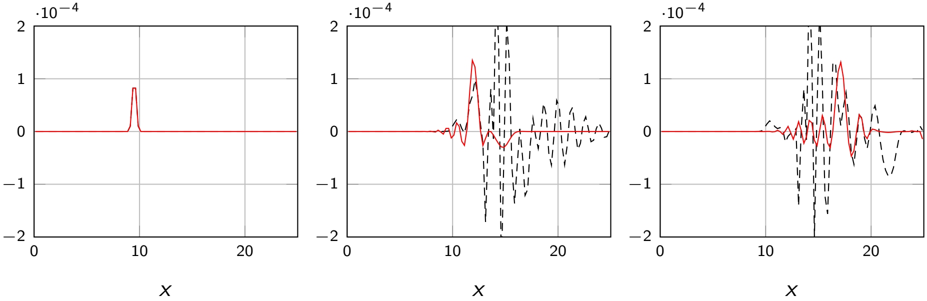


(a) WENO3

(b) WENO5

Figure: Subcritical flow: convergence tests with WENO3 and WENO5.

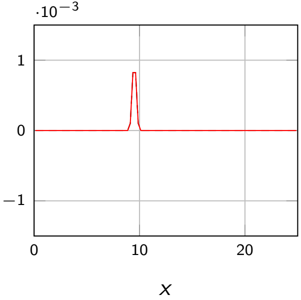
Validation: Small perturbation of supercritical flow without friction ($n = 0$)



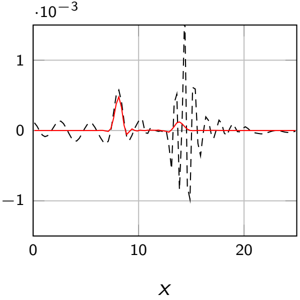
(a) $t = 0$ (b) $t = 0.33$ (c) $t = 1$

Figure: Perturbation on a subcritical flow: $\eta - \eta^{eq}$

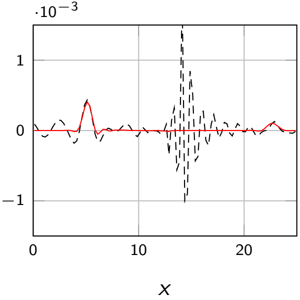
Validation: Small perturbation of subcritical flow without friction ($n = 0$)



(a) $t = 0$



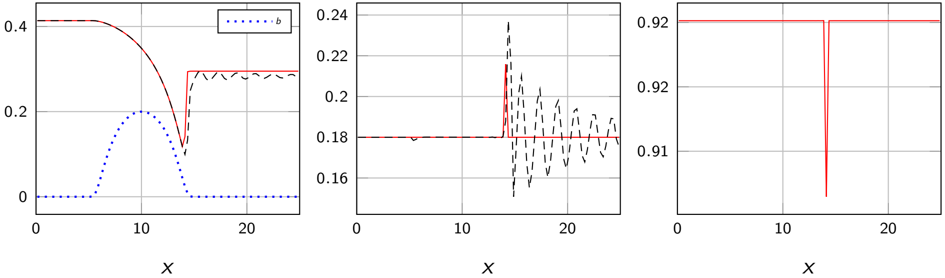
(b) $t = 0.66$



(c) $t = 2$

Figure: Perturbation on a supercritical flow: $\eta - \eta^{eq}$

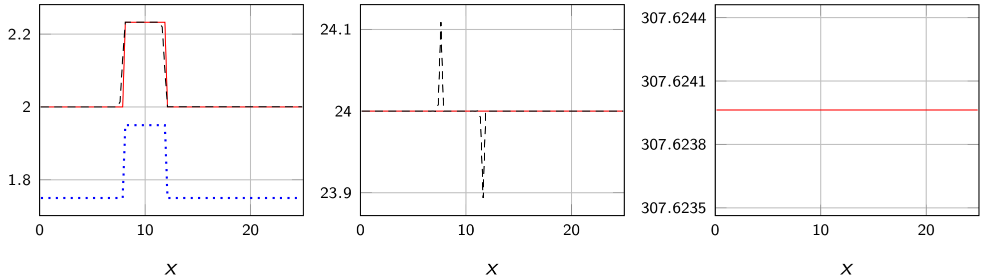
Validation: Discontinuous steady states without friction ($n = 0$)



(a) η (b) q (c) K

Figure: Transcritical flow: characteristic variables compute by means of the GF-WENO5 (red continuous line), WENO5 (black dashed line) schemes and b (blue continuous line) with $N_e = 100$.

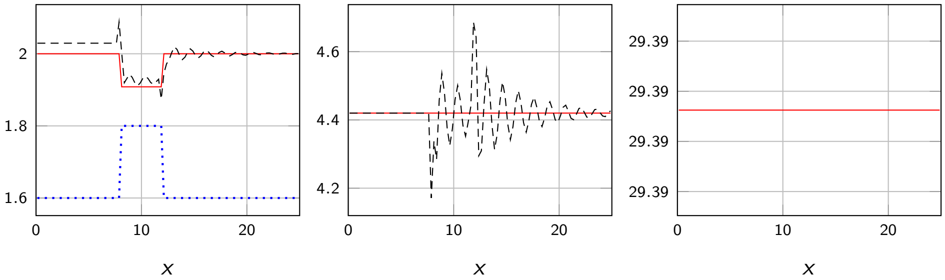
Validation: Discontinuous steady states without friction ($n = 0$)



(a) η (b) q (c) K

Figure: Supercritical flow: relevant variables computed with GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes and rescaled b (blue dotted line) with $N_e = 100$.

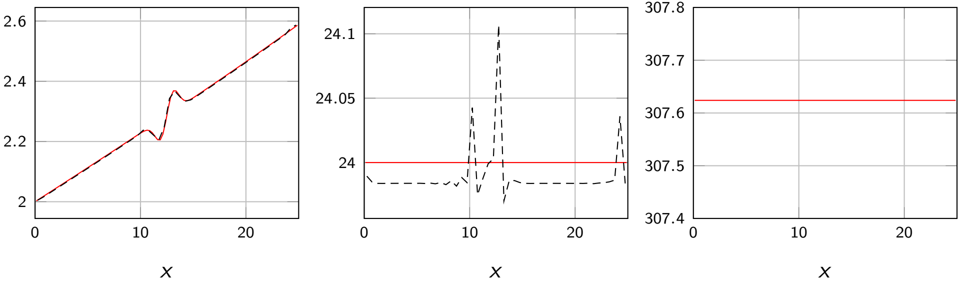
Validation: Discontinuous steady states without friction ($n = 0$)



(a) η (b) q (c) K

Figure: Subcritical flow: relevant variables computed with GF-WENO5 (red continuous line), WENO5 (black dashed line) schemes and rescaled b (blue dotted line) with $N_e = 100$.

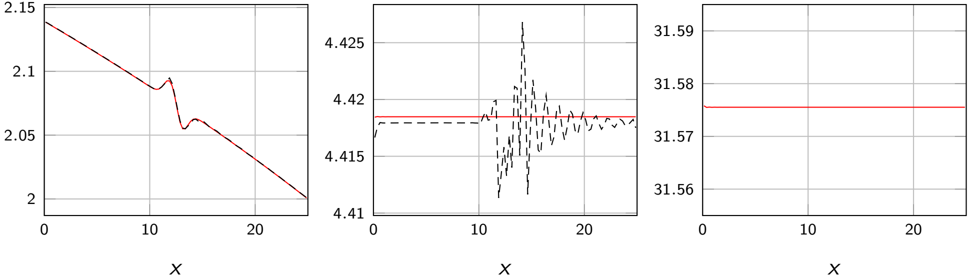
Validation: Steady states with friction ($n = 0.05$)



(a) η (b) q (c) K

Figure: Supercritical flow with friction: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes.

Validation: Steady states with friction ($n = 0.05$)



(a) η (b) q (c) K

Figure: Subcritical flow with friction: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes.

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Dispersive models[6]

BBM-KdV

$$\partial_t u+\partial_x f(u)+\mathcal{D}=0$$
$$\mathcal{D}=-\alpha\partial_{xxt}u+\beta\partial_{xxx}u$$

A Boussinesq system (Madsen and Søresen)

$$\begin{cases}\partial_t u+\partial_x q=0,\\ \partial_t q-\mathcal{T}^t[\partial_t q]+\partial_x(qu)+gh\partial_x u-\mathcal{T}^x[u]=0,\end{cases}$$
$$\mathcal{T}^t[\cdot]:=B_1\bar{h}^2\partial_{xx}[\cdot]+\frac{1}{3}\bar{h}\partial_x\bar{h}\partial_x[\cdot]$$
$$\mathcal{T}^x[\cdot]:=gB_2\bar{h}^2\left(2\partial_x\bar{h}+\bar{h}\partial_x\right)\partial_{xx}[\cdot],$$

Properties

- Both are some approximation of water waves
- Both includes dispersive terms (∂_{xxx})
- Both have some solitary waves as exact solution

$$\exists\varphi(x):\quad u(x,t)=\varphi(x-ct).$$

[6] with Wasilij Barsukow and Mario Ricchiuto

Global Flux for Dispersive equations

$$\partial_t u+\partial_x f(u) - \alpha\partial_{xxt} u + \beta\partial_{xxx} u = 0$$

Again, using the hypothesis $u(x,t) = \phi(x-ct)$, we can derive **a ""Global Flux"" recipe**.

Substituting $\partial_t = -c\partial_x$

$$\begin{aligned} 0 =& \partial_t u + f(u)_x - \alpha\partial_{xxt} u + \beta\partial_{xxx} u \\ =& - c\partial_x u + f(u)_x + (c\alpha + \beta)\partial_{xxx} u \end{aligned}$$

This suggests that exact traveling solutions preserve a global flux of the form

$$\mathcal{G} = -cu + f(u) + (c\alpha + \beta)\partial_{xx} u$$

Without replacing $\partial_t = -c\partial_x$

$$\mathcal{U} : \partial_x \mathcal{U} = \partial_t u \Longleftrightarrow \mathcal{U} = \mathcal{U}_0 + \int_{x_0}^{x} \partial_t u dx$$

$$\begin{aligned} &\partial_t u + \partial_x f(u) - \alpha\partial_{xxt} u + \beta\partial_{xxx} u = 0 \\ &\partial_x \mathcal{U} + \partial_x f(u) - \alpha\partial_{xxx}\mathcal{U} + \beta\partial_{xxx} u = 0 \\ &\partial_x \left((1-\alpha\partial_{xx})\mathcal{U} + f(u) + \beta\partial_{xx} u\right) = 0 \end{aligned}$$

Then, the **""Global Flux""**

$$\mathcal{G} := (1-\alpha\partial_{xx})\mathcal{U} + f(u) + \beta\partial_{xx} u.$$

Discrete Global Flux for Dispersive equations

$$\partial_x \left((1-\alpha\partial_{xx})\mathcal{U} + f(u) + \beta\partial_{xx} u\right)$$

For a soliton...

Discrete operators

$$\mathcal{I} \approx \int_{x_0}^{x} \qquad \mathbb{M} = \mathbb{D}_1\mathcal{I}$$

$$\mathbb{D}_1 \approx \partial_x \qquad \mathbb{D}_1\mathcal{U} = \mathcal{I}\partial_t u$$

$$\mathbb{D}_2 \approx \partial_{xx}$$

Method

$$\mathbb{D}_1\left((1-\alpha\mathbb{D}_2)\mathcal{U} + f(u) + \beta\mathbb{D}_2 u\right) = 0$$

$$(1-\alpha\mathbb{D}_2)\mathbb{D}_1\mathcal{I}\partial_t u + \mathbb{D}_1\left(f(u) + \beta\mathbb{D}_2 u\right) = 0$$

Assumption on IC

$$\mathbb{D}_1\left(f(u) + \beta\mathbb{D}_2 u\right) = -c(1-\alpha\mathbb{D}_2)\mathbb{D}_1 u$$

In the method we obtain...

$$\underbrace{(1-\alpha\mathbb{D}_2)}_{>0}\left(\mathbb{D}_1\mathcal{I}\partial_t u - c\mathbb{D}_1 u\right) = 0$$

$$\mathbb{M}\partial_t u - c\mathbb{D}_1 u = 0$$

Transport equation!

Discrete Global Flux Accuracy

$$\mathbb{M}\partial_t u - c\mathbb{D}_1 u = 0 \tag{1}$$

Choose mass first

For a given $\mathbb{M}$ can we optimize $\mathbb{D}_1$ such that $\mathbb{M}^{-1}\mathbb{D}_1$ is a higher order operator?

$$\begin{cases} \mathbb{M} = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right) \\ \mathbb{D}_1 = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)\frac{1}{\Delta x} \qquad \text{(FEM)}, \\ O(\Delta x^4) \end{cases}$$

$$\begin{cases} \mathbb{M} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \\ \mathbb{D}_1 = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)\frac{1}{\Delta x} \qquad \text{(GF) (from RD)} \\ O(\Delta x^2) \end{cases}$$

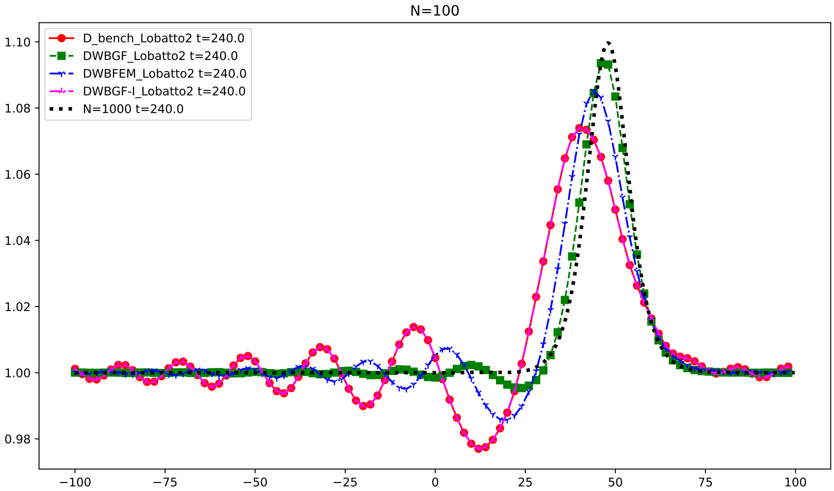
Choose derivative First

Not today

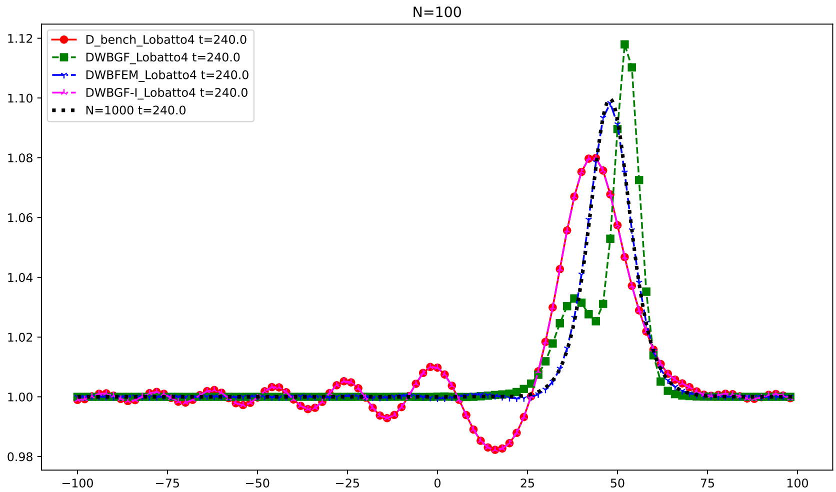
Benchmark and other operators

$$\begin{cases} \mathbb{M} = 1 \\ \mathbb{D}_1 = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)\frac{1}{\Delta x} \qquad \text{(Dbench)} \\ \mathbb{D}_2 = (1, -2, 1)\frac{1}{\Delta x^2} \\ \mathbb{D}_3 = \left(-\frac{1}{2}, 1, 0, -1, \frac{1}{2}\right)\frac{1}{\Delta x^3} \end{cases}$$

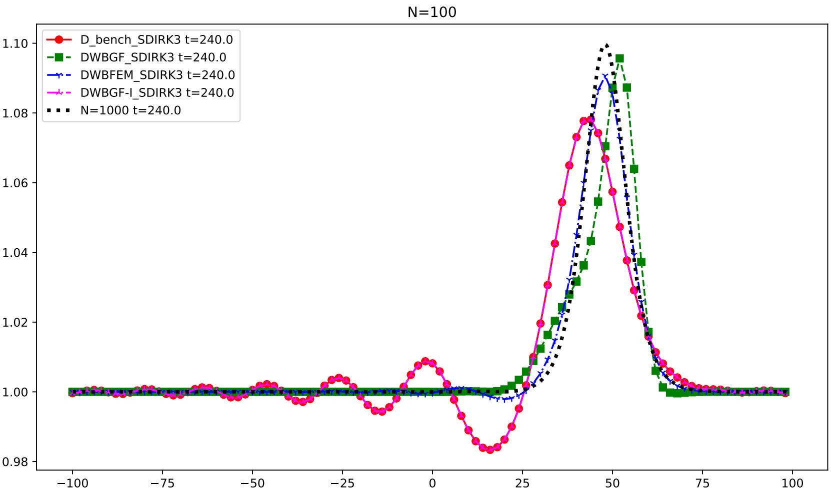
Test on soliton: Lobatto IIIA 2nd order, CFL 1.2



Test on soliton: Lobatto IIIA 4nd order, CFL 1.2



Test on soliton: SDIRK3, CFL 1.2



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Conclusion

Global Flux FV for SW

- Formulation in $\mathcal{G}$
- Well balanced for LAR and moving equilibria
- WENO + DeC $\Longrightarrow$ Arbitrarily High order
- Intrinsically 1D method
- 2D extension on Cartesian grids
- Ciallella, Torlo, Ricchiuto `https://arxiv.org/abs/2205.13315`

Other GF applications

- Dispersive Waves (some preliminary results)
  - Connection to SBP operators?
- Other residual to balance
  - Divergence free schemes
  - Low Mach/Low Froude schemes
  - IMEX versions...

Conclusion

Global Flux FV for SW

- Formulation in $\mathcal{G}$
- Well balanced for LAR and moving equilibria
- WENO + DeC $\Longrightarrow$ Arbitrarily High order
- Intrinsically 1D method
- 2D extension on Cartesian grids
- Ciallella, Torlo, Ricchiuto `https://arxiv.org/abs/2205.13315`

Other GF applications

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THANK YOU!