IMEX ADER and DeC: arbitrary high order schemes, stability and application to advection–diffusion–dispersion

Davide Torlo, Philipp Öffner, Louis Petri, Maria Han Veiga, Lorenzo Micalizzi

SISSA MathLab, Mathematics Area, SISSA International School for Advanced Studies, Trieste, Italy

INDAM Workshop INSIDEs

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Rome - 21st February 2024

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Outline

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DeC (Deferred Correction)

- Originally Nonlinear Solver ('60)
- Spectral formulation ODE solver: explicit (Dutt et al. 2000), implicit/IMEX (Minion 2003)
- More general operators for PDEs (Abgrall 2018)
- Arbitrary high order method (ODE/PDE)
- High Order FEM discretization in time
- Explicit, Implicit, IMEX
- Two operators
- Iterative method

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ADER (Arbitrary Derivatives)

- High order discretization for PDEs through Cauchy–Kovalevskaya (Titarev, Toro 2002)
- High order DG in space-time (Dumbser et al. 2008)
- Arbitrary high order methods (PDE)
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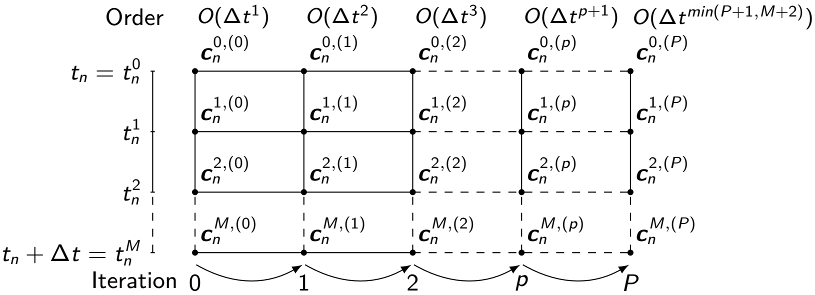
Relationship between ADER and DeC as ODE solvers (Han Veiga et al. 2020)

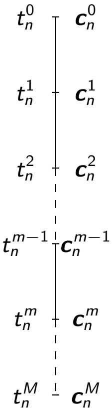
DeC and ADER: arbitrary high order methods

DeC/ADER discretization and iterations

$$\boldsymbol{c}(t_n) \approx \boldsymbol{c}_n \qquad \boldsymbol{c}(t) = \sum_{m=0}^{M} \varphi_n^m(t)\boldsymbol{c}_n^m \quad t \in [t_n, t_{n+1}] \Longrightarrow \boldsymbol{c}_{n+1} \approx \boldsymbol{c}(t_{n+1})$$

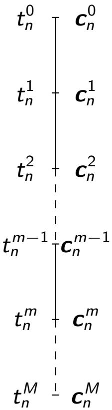


DeC and ADER for ODEs [1,2]



$$\frac{d}{dt}\boldsymbol{c}(t) = \mathbf{G}(t, \boldsymbol{c}(t)), \qquad \boldsymbol{c}_n \approx \boldsymbol{c}(t_n), \qquad \boldsymbol{c}(t) = \sum_{m=0}^{M} \varphi_n^m(t)\boldsymbol{c}_n^m \qquad \forall t \in [t_n, t_{n+1}]$$

[1] M. Han Veiga, P. Öffner, and D. T.. "DeC and ADER: Similarities, Differences and a Unified Framework." JSC, 87, 2 (2021)
[2] M. Han Veiga, L. Micalizzi and D. T.. "On improving the efficiency of ADER methods." AMC, 466, page 128426, (2024)



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DeC high order operator

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^m - \boldsymbol{c}_n - \int_{t_n^0}^{t_n^m} \mathbf{G}(\boldsymbol{c}(t))dt = 0 \quad \forall m \in [\![1, M]\!]$$

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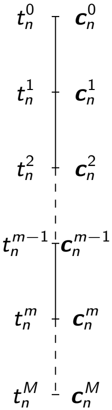

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$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^m - \boldsymbol{c}_n - \Delta t \sum_{r=0}^{M} \theta_r^m \mathbf{G}(\boldsymbol{c}_n^r) = 0 \quad \forall m \in [\![1, M]\!]$$

- Based on integral formulation
- Collocation methods
- Implicit RK with full A
- Difficult to solve directly

- Choice on points
- Gauss–Lobatto $\Longrightarrow$ Lobatto IIIA
- High order of accuracy
  - for Lobatto $2M$

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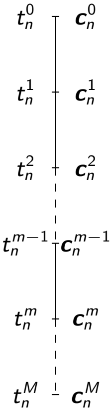

$$\frac{d}{dt}\boldsymbol{c}(t) = \mathbf{G}(t,\boldsymbol{c}(t)), \qquad \boldsymbol{c}_n \approx \boldsymbol{c}(t_n), \qquad \boldsymbol{c}(t) = \sum_{m=0}^{M} \varphi_n^m(t)\boldsymbol{c}_n^m \qquad \forall t \in [t_n, t_{n+1}]$$

ADER high order operator

$$\forall m \in [\![0, M]\!],$$

$$\int_{t_n}^{t_{n+1}} \varphi_n^m(t)\partial_t \boldsymbol{c}(t)dt - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\mathbf{G}(\boldsymbol{c}(t))dt = 0$$

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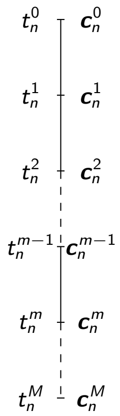

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$$\forall m \in [\![0, M]\!], \quad \mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := A^{m,r}\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - R^{m,r}\mathbf{G}(\boldsymbol{c}_n^r) =$$

$$\varphi_n^m(t_{n+1})\varphi_n^r(t_{n+1})\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - \int_{t_n}^{t_{n+1}} \partial_t\varphi_n^m(t)\varphi_n^r(t)dt\, \boldsymbol{c}_n^r - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\varphi_n^r(t)dt\, \mathbf{G}(\boldsymbol{c}_n^r) = 0$$

- Based on weak formulation
- Integration by parts
- Implicit RK with full A
- Difficult to solve directly

- Gauss–Lobatto $\Longrightarrow$ Lobatto IIIC
- High order of accuracy
  - for Lobatto $2M$
  - for Gauss–Legendre $2M+1$

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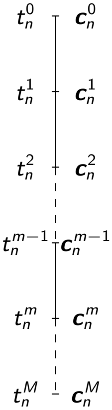
Examples of $\mathcal{L}^2$



DeC $\mathcal{L}^2$ operator

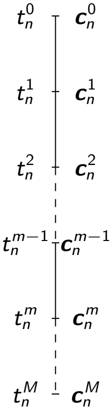
$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^m - \boldsymbol{c}_n^0 - \Delta t \sum_{r=0}^{M} \theta_r^m \mathbf{G}(\boldsymbol{c}_n^r) = 0 \quad \forall m \in [\![1, M]\!]$$

$M+1=2$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^1 - \boldsymbol{c}_n^0 - \Delta t \frac{1}{2}\left(\mathbf{G}(\boldsymbol{c}_n^0) + \mathbf{G}(\boldsymbol{c}_n^1)\right)$$

0		
1	$\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$

Examples of $\mathcal{L}^2$



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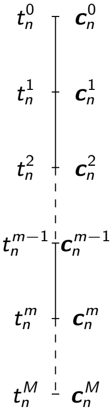
$$\begin{array}{c|cc} 0 & & \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$M+1=3$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2)\right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2)\right) \end{pmatrix}$$

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Examples of $\mathcal{L}^2$



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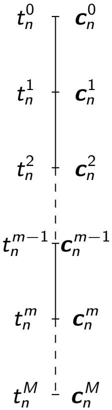
$$\forall m \in [\![0, M]\!], \quad \mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := A^{m,r}\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - R^{m,r}\mathbf{G}(\boldsymbol{c}_n^r) =$$

$$\varphi_n^m(t_{n+1})\varphi_n^r(t_{n+1})\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - \int_{t_n}^{t_{n+1}} \partial_t\varphi_n^m(t)\varphi_n^r(t)dt\, \boldsymbol{c}_n^r - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\varphi_n^r(t)dt\, \mathbf{G}(\boldsymbol{c}_n^r) = 0$$



ADER $\mathcal{L}^2$ operator

$$\forall m \in [\![0, M]\!], \quad \mathcal{L}^{2,m}(\underline{\mathbf{c}}) := \mathbf{c}_n^m - \mathbf{c}_n - (A^{-1})_{m,\ell} R^{\ell,r} \mathbf{G}(\mathbf{c}_n^r) = 0$$

$M+1=2$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\mathbf{c}}) = \begin{pmatrix} \mathbf{c}_n^0 - \mathbf{c}_n - \Delta t \left(\frac{1}{2}\mathbf{G}(\mathbf{c}_n^0) - \frac{1}{2}\mathbf{G}(\mathbf{c}_n^1) \right) \\ \mathbf{c}_n^1 - \mathbf{c}_n - \Delta t \left(\frac{1}{2}\mathbf{G}(\mathbf{c}_n^0) + \frac{1}{2}\mathbf{G}(\mathbf{c}_n^1) \right) \end{pmatrix}$$

0	$-\frac{1}{2}$	$\frac{1}{2}$
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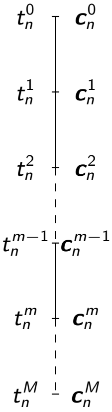
Examples of $\mathcal{L}^2$



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$M+1=2$ **Gauss–Lobatto nodes**

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^0 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) - \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^1) \right) \\ \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^1) \right) \end{pmatrix}$$

$$\begin{array}{c|cc} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$M+1=3$ **Gauss–Lobatto nodes**

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^0 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) - \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{5}{12}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{12}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix}$$

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

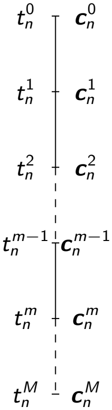
Properties of $\mathcal{L}^2 = 0$

Method	DeC		ADER		
Nodes	Equispaced	Gauss–Lobatto	Equispaced	Gauss–Lobatto	Gauss–Legendre
Order	$M+1$	$2M$	$M+1$	$2M$	$2M+1$ [3]
Known method	Collocation	Lobatto IIIA		Lobatto IIIC	
A–stability	☹	☺	☹	☺	☺ [4]

[3] M. Han Veiga, L. Micalizzi and D. T.. "On improving the efficiency of ADER methods." AMC, 466, page 128426, (2024)

[4] P. Öffner, L. Petri, D.T.. "Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations" (2024)

DeC and ADER operators



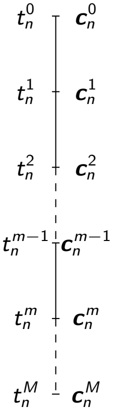
DeC operators

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := \boldsymbol{c}_n^m - \boldsymbol{c}_n^0 - \sum_{r=0}^{M} \int_{t_n^0}^{t_n^m} \varphi_n^r(t) dt \, \mathbf{G}(\boldsymbol{c}_n^r) = 0, \qquad \forall m \in [\![1, M]\!],$$

ADER operators

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := A^{m,r} \boldsymbol{c}_n^r - \varphi_n^m(t_n) \boldsymbol{c}_n - \int_{t_n}^{t_{n+1}} \varphi_n^m(t) \varphi_n^r(t) dt \, \mathbf{G}(\boldsymbol{c}_n^r) = 0, \qquad \forall m \in [\![0, M]\!],$$

DeC and ADER operators



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$$\mathcal{L}^{1,m}(\underline{\mathbf{c}}) := \mathbf{c}_n^m - \mathbf{c}_n^0 - \int_{t_n^0}^{t_n^m} 1 dt\, \mathbf{G}(\mathbf{c}_n) = 0, \qquad \forall m \in [\![1, M]\!].$$

ADER operators

$$\mathcal{L}^{2,m}(\underline{\mathbf{c}}) := A^{m,r}\mathbf{c}_n^r - \varphi_n^m(t_n)\mathbf{c}_n - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\varphi_n^r(t) dt\, \mathbf{G}(\mathbf{c}_n^r) = 0, \qquad \forall m \in [\![0, M]\!],$$

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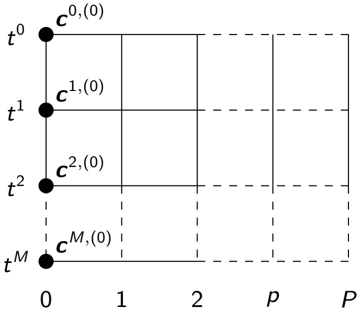
Deferred Correction Iterative procedure

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

- $\mathcal{L}^1(\underline{\boldsymbol{c}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{c}}) = 0$, high order $Q(= 2M)$.

$$\boldsymbol{c}^{m,(0)} := \boldsymbol{c}(t_n), \quad m = 0, \dots, M$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$



DeC Theorem

- $\mathcal{L}^1$ coercive with constant $\mathcal{O}(1)$
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

DeC converges and $\min(P, Q)$ is the order of accuracy.

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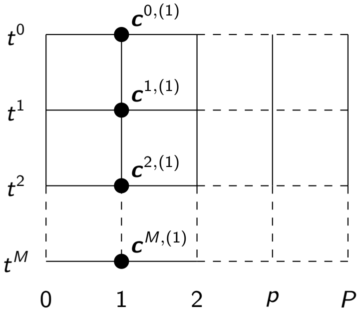

DeC Theorem

- $\mathcal{L}^1$ coercive with constant $\mathcal{O}(1)$
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

DeC converges and $\min(P, Q)$ is the order of accuracy.

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$$\boldsymbol{c}^{m,(0)} := \boldsymbol{c}(t_n), \quad m = 0, \dots, M$$
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DeC Theorem

- $\mathcal{L}^1$ coercive with constant $\mathcal{O}(1)$
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

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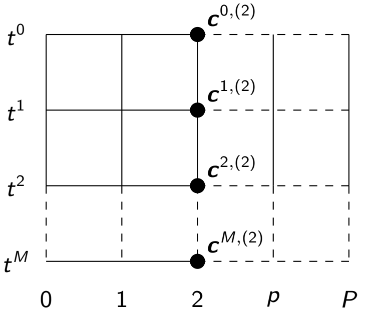

Deferred Correction Iterative procedure

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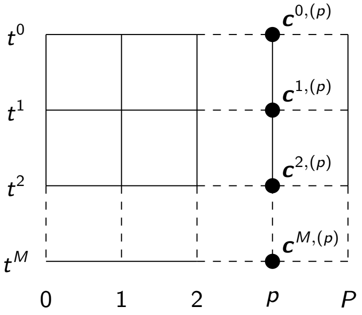
- $\mathcal{L}^1(\underline{\mathbf{c}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\mathbf{c}}) = 0$, high order $Q(= 2M)$.

$$\mathbf{c}^{m,(0)} := \mathbf{c}(t_n), \quad m = 0, \dots, M$$
$$\mathcal{L}^1(\underline{\mathbf{c}}^{(p)}) = \mathcal{L}^1(\underline{\mathbf{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\mathbf{c}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$



DeC Theorem

- $\mathcal{L}^1$ coercive with constant $\mathcal{O}(1)$
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

DeC converges and $\min(P, Q)$ is the order of accuracy.

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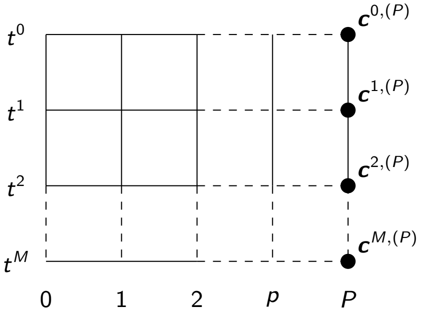

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DeC converges and $\min(P, Q)$ is the order of accuracy.

Example of explicit DeC $M = 2$ $P = 3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix}$$

Example of explicit DeC $M=2$ $P=3$

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$$*\boldsymbol{c}_n^{(1),1} - \boldsymbol{c}_n^{(1),0} - \Delta t \mathbf{G}(\boldsymbol{c}_n^{(1),0}) = \boldsymbol{c}_n^{(0),1} - \boldsymbol{c}_n^{(0),0} - \Delta t \mathbf{G}(\boldsymbol{c}_n^{(0),0}) -$$
$$\boldsymbol{c}_n^{(0),1} + \boldsymbol{c}_n^{(0),0} + \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^{(0),0}) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^{(0),1}) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^{(0),2}) \right)$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2)\right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2)\right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

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$$\begin{array}{c|c} 0 & \\ \frac{1}{2} & \frac{1}{2} \\ & \\ & \\ \hline & \end{array}$$

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$$\begin{array}{c|c} 0 & \\ \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \\ & \\ \hline & \end{array}$$

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0			
$\frac{1}{2}$	$\frac{1}{2}$		
1	1		
$\frac{1}{2}$	$\frac{5}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
1	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

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$$*\boldsymbol{c}_{n+1} = \boldsymbol{c}_n^{(3),2} = \boldsymbol{c}_n + \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(2),0}) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^{(2),1}) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(2),2}) \right)$$

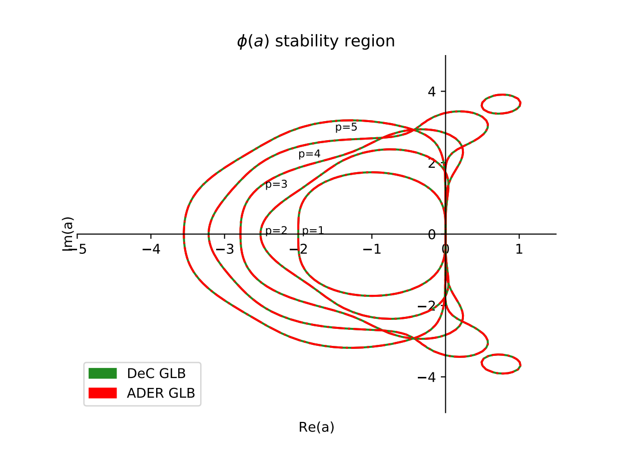
$$\begin{array}{c|ccccc} 0 & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ 1 & 1 & & & & \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} & & \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ \hline & \frac{1}{6} & & & \frac{2}{3} & \frac{1}{6} \end{array}$$

Stability of explicit DeC/ADER

Stability function

All the described DeC/ADER explicit methods of order P have stability function given by

$$R(z) = \sum_{r=0}^{P} \frac{1}{r!} z^r.$$



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Implicit DeC/ADER

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- $\mathcal{L}^1$ implicit
- Fully implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \mathbf{G}(\underline{\boldsymbol{c}})$$

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Implicit DeC/ADER

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- Linearly implicit

$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \left(\mathbf{G}(\mathbf{c}_n) + \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t A^{-1} R \left(\mathbf{G}(\mathbf{c}_n) + \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$

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Implicit DeC/ADER

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DeC Full Implicit IMDeC

$$\underline{\mathbf{c}}^{(p)} - \underline{\mathbf{c}}^{(p-1)} + \Delta t \beta \left(\mathbf{G}(\underline{\mathbf{c}}^{(p)}) - \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})\right)$$
$$= \mathbf{c}_n - \underline{\mathbf{c}}^{(p-1)} + \Delta t \Theta \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})$$

DeC Linearly Implicit IMDeC-Lin

$$\left[I - \Delta t \beta \partial_c \mathbf{G}(\mathbf{c}_n)\right] \left(\underline{\mathbf{c}}^{(p)} - \underline{\mathbf{c}}^{(p-1)}\right)$$
$$= \mathbf{c}_n - \underline{\mathbf{c}}^{(p-1)} + \Delta t \Theta \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})$$

Implicit DeC/ADER

$\mathcal{L}^1(\underline{c}^{(p)}) = \mathcal{L}^1(\underline{c}^{(p-1)}) - \mathcal{L}^2(\underline{c}^{(p-1)})$

Implicit R

- $\mathcal{L}^1$ imp
- Fully i

$\mathbf{G}(\underline{c}^{(p-1)}))$

- Linearl

-Lin

$\mathcal{L}^1(\underline{c})$

$\mathcal{L}^1(\underline{c})$

$\underline{c}^{(p-1)})$

$= \underline{c}_n - \underline{c} + \Delta t \Theta \mathbf{G}(\underline{c}^{(p-1)})$

This leads to the following RK Butcher tableau

$$
\begin{array}{c|cccccccc}
0 & 0 & & & & & & & \\
\underline{\beta} & \underline{\underline{0}} & \underline{\underline{B}} & & & & & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & & & & & \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & & & & \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & \\
1 & \underline{\theta}_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \underline{\beta}^M \\
\hline
 & \underline{\theta}_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \underline{\beta}^M
\end{array},
$$

with $B_{mr} = \delta_{mr}\beta^m$ for $m, r = 1, \dots, M$ and δ_{mr} the Kronecker delta.

Implicit DeC/ADER

$$\mathcal{L}^1(\underline{\mathbf{c}}^{(p)}) = \mathcal{L}^1(\underline{\mathbf{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\mathbf{c}}^{(p-1)})$$

Implicit Recipe

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$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \mathbf{G}(\underline{\mathbf{c}})$$
$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}})$$

- Linearly implicit

$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \left(\mathbf{G}(\mathbf{c}_n) + \partial_c \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$
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ADER Full Implicit IMADER

$$\mathcal{L}^1 = \mathcal{L}^2$$
$$\underline{\mathbf{c}}^{(p)} - \mathbf{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}}^{(p)}) = 0$$

ADER Linearly Implicit IMADER-Lin

$$\left[I - \Delta t A^{-1} R \partial_c \mathbf{G}(\mathbf{c}_n)\right] (\underline{\mathbf{c}}^{(p)} - \underline{\mathbf{c}}^{(p-1)})$$
$$= \mathbf{c}_n - \underline{\mathbf{c}}^{(p-1)} + \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})$$

Implicit DeC/ADER

$$
\begin{array}{c|cccccc}
\underline{P} & \underline{\underline{Q}} & & & & & \\
\underline{P} & \underline{\underline{0}} & \underline{\underline{Q}} & & & & \\
\vdots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & & \\
\vdots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\underline{P} & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\hline
 & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{b}^T
\end{array}
$$

$\boldsymbol{c}_n))$

$\underline{\boldsymbol{c}} - \boldsymbol{c}_n))$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

ADER Full Implicit IMADER

$$\mathcal{L}^1 = \mathcal{L}^2$$

$$\underline{\boldsymbol{c}}^{(p)} - \boldsymbol{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\boldsymbol{c}}^{(p)}) = 0$$

ADER Linearly Implicit IMADER-Lin

$$\left[I - \Delta t A^{-1} R \partial_{\boldsymbol{c}} \mathbf{G}(\boldsymbol{c}_n) \right] (\underline{\boldsymbol{c}}^{(p)} - \underline{\boldsymbol{c}}^{(p-1)})$$

$$= \boldsymbol{c}_n - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t A^{-1} R \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})$$

Example of IMDeC and IMDeC-Lin

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c})$$

IMDeC2

$$*\boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}^{(1),0} = \boldsymbol{c}^{(2),0} = \boldsymbol{c}_n$$

$$*\boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \Delta t \mathbf{G}(\boldsymbol{c}^{(1),1})$$

$$*\boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \Delta t \left(\mathbf{G}(\boldsymbol{c}^{(2),1}) - \mathbf{G}(\boldsymbol{c}^{(1),1}) + \frac{\mathbf{G}(\boldsymbol{c}^{(1),1}) + \mathbf{G}(\boldsymbol{c}^{(1),0})}{2} \right)$$

IMDeC2-Lin

$$*\boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}^{(1),0} = \boldsymbol{c}^{(2),0} = \boldsymbol{c}_n$$

$$*\boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \Delta t \partial_{\boldsymbol{c}} \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(1),1} - \boldsymbol{c}_n) + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$*\boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \Delta t \left(\partial_{\boldsymbol{c}} \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(2),1} - \boldsymbol{c}^{(1),1}) + \frac{\mathbf{G}(\boldsymbol{c}^{(1),1}) + \mathbf{G}(\boldsymbol{c}^{(1),0})}{2} \right)$$

Example of IMADER and IMADER-Lin

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c})$$

IMADER2

$$*\boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}_n$$

$$*\boldsymbol{c}^{(1),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}(-\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1}))$$

$$*\boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}(\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1}))$$

$$*\boldsymbol{c}^{(2),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}(-\mathbf{G}(\boldsymbol{c}^{(2),0}) + \mathbf{G}(\boldsymbol{c}^{(2),1}))$$

$$*\boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}(\mathbf{G}(\boldsymbol{c}^{(2),0}) + \mathbf{G}(\boldsymbol{c}^{(2),1}))$$

Useless!

IMADER2-Lin

$$*\boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}_n$$

$$*\boldsymbol{c}^{(1),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(-\boldsymbol{c}^{(1),0} + \boldsymbol{c}^{(1),1})$$

$$*\boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(1),0} + \boldsymbol{c}^{(1),1} - 2\boldsymbol{c}_n) + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$*\boldsymbol{c}^{(2),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(-\boldsymbol{c}^{(2),0} + \boldsymbol{c}^{(2),1} + \boldsymbol{c}^{(1),0} - \boldsymbol{c}^{(1),1}) + \frac{\Delta t}{2}\left(-\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1})\right)$$

$$*\boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(2),0} + \boldsymbol{c}^{(2),1} - \boldsymbol{c}^{(1),0} - \boldsymbol{c}^{(1),1}) + \frac{\Delta t}{2}\left(\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1})\right)$$

A-Stable?

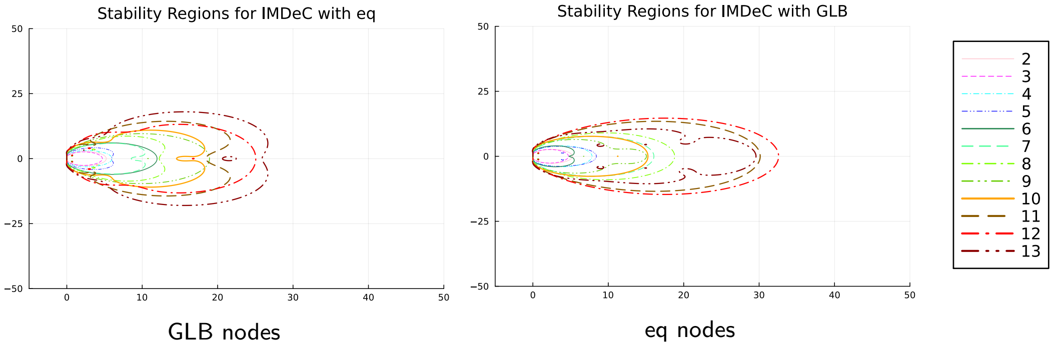


Figure: ImDeC stability region for orders 2 to 13.

Stability of IMDeC

Almost A-Stable!

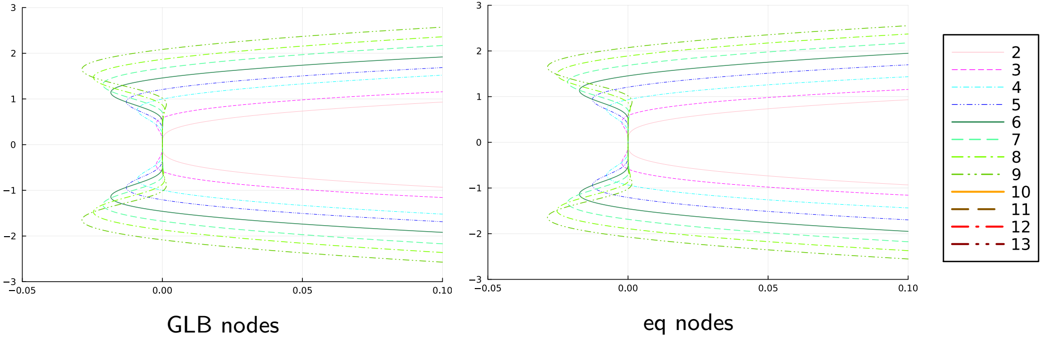
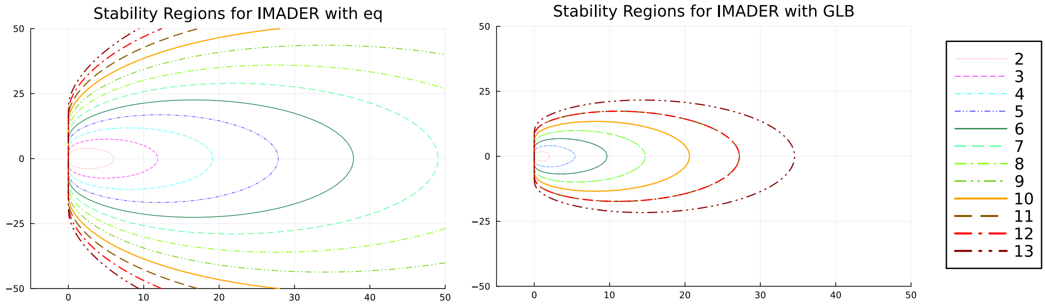


Figure: Zoomed ImDeC stability region for orders 2 to 7.

Stability of IMADER

A-Stable? GLB Yes! Proof [5], Equi Not clear



GLB nodes

eq nodes

Figure: ImADER stability region for orders 2 to 13.

[5] P. Öffner, L. Petri, D.T.. "Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations" (2024)

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S} \cdot \boldsymbol{c}$$

IMEX Recipe

- $\mathcal{L}^1$ implicit for $\mathbf{S}$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S} \cdot \boldsymbol{c}$$

IMEX Recipe

- $\mathcal{L}^1$ implicit for $\mathbf{S}$
- Nonlinear implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

IMEX DeC and ADER

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- $\mathcal{L}^1$ implicit for $\mathbf{S}$
- Nonlinear implicit

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$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

- Linearly IMEX (EIN methods / Add-and-subtract)

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S} \cdot \boldsymbol{c}$$

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- $\mathcal{L}^1$ implicit for $\mathbf{S}$
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$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

- Linearly IMEX (EIN methods / Add-and-subtract)

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_c \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_c \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

IMEX DeC (nonlinear)

$$\underline{\boldsymbol{c}}^{(p)} - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}^{(p)}) - \mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)})\right)$$
$$= \boldsymbol{c}_n - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t \Theta(\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)}))$$
$$\Longleftrightarrow$$
$$\underline{\boldsymbol{c}}^{(p)} = \boldsymbol{c}_n + \Delta t \Big[\beta \mathbf{S}(\underline{\boldsymbol{c}}^{(p)})$$
$$+ (\Theta - \beta)\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \Theta \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\Big]$$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}\cdot \boldsymbol{c}$$

$$\begin{array}{c|cccccccc}
0 & 0 \\
\underline{\beta} & \underline{0} & \underline{\underline{B}} \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
1 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \beta^M \\
\hline
 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \beta^M
\end{array}, \qquad
\begin{array}{c|cccccccc}
0 & 0 \\
\underline{\beta} & \underline{\beta} \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} \\
1 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\theta}_r^M & 0 \\
\hline
 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\theta}_r^M & 0
\end{array}.$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\underline{\boldsymbol{c}}^{(p)} = \boldsymbol{c}_n + \Delta t \Big[\beta \mathbf{S}(\underline{\boldsymbol{c}}^{(p)})$$
$$+ (\Theta - \beta)\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \Theta \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\Big]$$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}\cdot \boldsymbol{c}$$

IMEX Recipe

- $\mathcal{L}^1$ implicit for $\mathbf{S}$
- Nonlinear implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

- Linearly IMEX (EIN methods / Add-and-subtract)

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_c \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_c \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

IMEX ADER (nonlinear)

$$\underline{\boldsymbol{c}}^{(p)} - \underline{\boldsymbol{c}}^{(p-1)} - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}^{(p)}) - \mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)})\right)$$
$$= \boldsymbol{c}_n - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\right)$$
$$\iff$$
$$\underline{\boldsymbol{c}}^{(p)} = \boldsymbol{c}_n + \Delta t A^{-1} R \left[\mathbf{S}(\underline{\boldsymbol{c}}^{(p)}) + \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\right]$$

IMEX DeC and ADER

$$\partial_t c = \mathbf{G}(c) + \mathbf{S}(c) \text{ or better } \partial_t c = \mathbf{G}(c) + \mathbf{S}_c c$$

$$\begin{array}{c|cccccccc}
0 & 0 \\
\underline{P} & \underline{0} & \underline{\underline{Q}} \\
\underline{P} & \underline{0} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\underline{P} & \underline{0} & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\hline
 & \underline{0}^T & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{b}^T
\end{array},
\qquad
\begin{array}{c|ccccccc}
0 & 0 \\
\underline{P} & \underline{P} \\
\underline{P} & \underline{0} & \underline{\underline{Q}} \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\underline{P} & \underline{0} & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{Q}} \\
\hline
 & \underline{0}^T & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{b}^T
\end{array}.$$

$$\mathcal{L}^1(\underline{c}) := \underline{c} - c_n - \Delta t A^{-1} R \left(\partial_c \mathbf{S}(c_n)\underline{c} + \mathbf{G}(c_n)\right)$$

Stability for IMEX methods

How to compute the stability region for IMEX methods? $\partial_t \mathbf{c} = G\mathbf{c} + S\mathbf{c}, \quad G, S \in \mathbb{C}$

$\mathbf{c}_{n+1} = R(\Delta t G, \Delta t S)\mathbf{c}_n = R(\lambda_G, \lambda_S)\mathbf{c}_n \qquad R(\cdot,\cdot) : \mathbb{C}^2 \to \mathbb{C} \qquad$ Hard to study $\{|R| \leq 1\} \subset \mathbb{C}^2$

Minion[a]

- $\lambda_G \in i\mathbb{R}$
- $\lambda_S \in \mathbb{R}$
- $R(\lambda_G, \lambda_S) : \mathbb{C} \to \mathbb{C}$
- Not really representative of high order operators
- Simple for comparisons

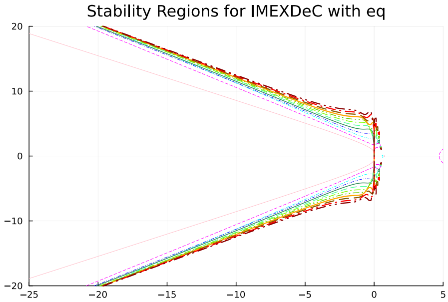
[a]M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. Commun. Math. Sci., 1(3):471–500, 09 2003.

Hundsdorfer[a]

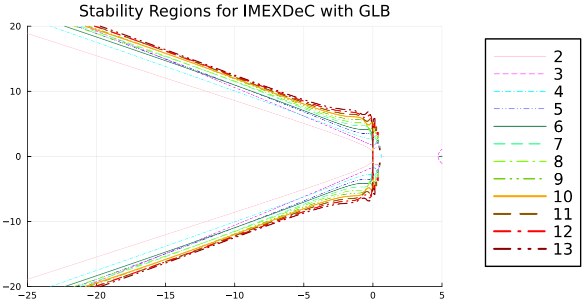
- $\mathcal{D}_0 := \{\lambda_G \in \mathbb{C} : |R(\lambda_G, \lambda_S)| \leq 1, \forall \lambda_S \in \mathbb{C}^-\}$
- $\mathcal{D}_1 := \{\lambda_S \in \mathbb{C} : |R(\lambda_G, \lambda_S)| \leq 1, \forall \lambda_G \in \mathcal{S}_0\}$
  - $\mathcal{S}_0 = \{z \in \mathbb{C} : |1 + z| \leq 1\}$
- Quite restrictive
  - $\mathcal{D}_0 = \emptyset$ often, we are asking essentially more than A–stability
- Numerical discretization more involved than Minion's one

[a]W. Hundsdorfer and J. Verwer. Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations. Springer Berlin Heidelberg, 2003.

IMEX DeC Stability Region with Minion's approach



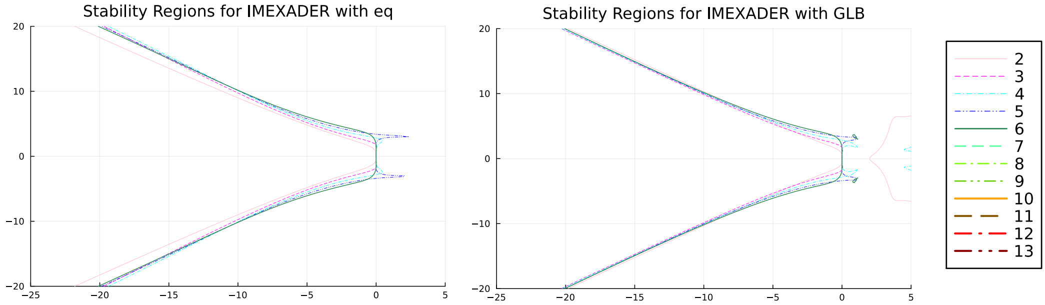
Equispaced nodes



Gauss–Lobatto nodes

Minion's Approach

IMEX ADER Stability Region with Minion's approach

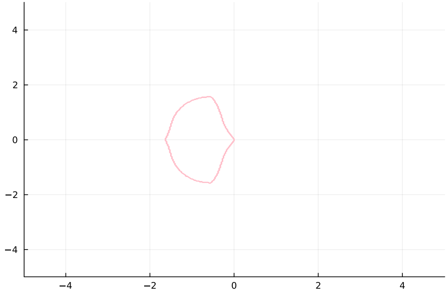


Equispaced nodes

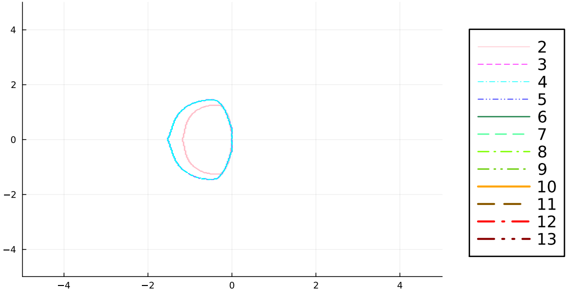
Gauss–Lobatto nodes

Hundsdorfer's Approach

IMEX ADER Stability Region with $\mathcal{D}_0$ Hundsdorfer's approach

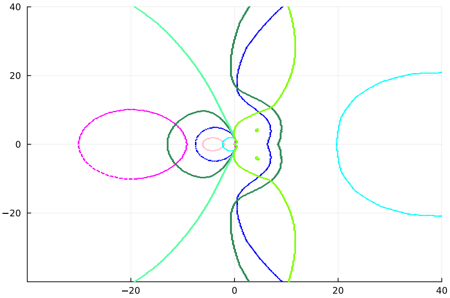


Equispaced nodes for order 2

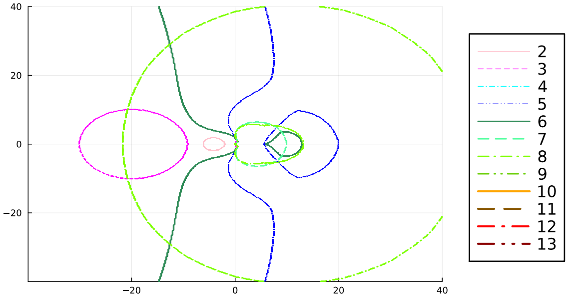


Gauss–Lobatto nodes for orders 2 to 4

IMEX DeC Stability Region with $\mathcal{D}_1$ Hundsdorfer's approach: Bounded areas



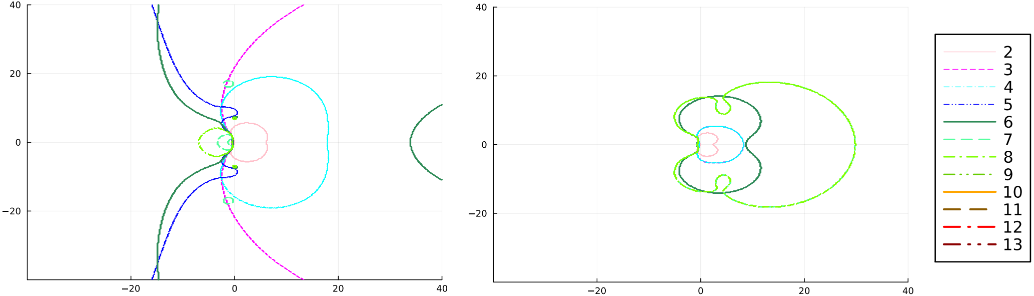
Equispaced nodes



Gauss–Lobatto nodes

Hundsdorfer's Approach

IMEX ADER Stability Region with $\mathcal{D}_1$ Hundsdorfer's approach: Unbounded areas



Equispaced nodes   Gauss–Lobatto nodes

IMEX Stability Summary

Method	Minion	$\mathcal{D}_0$ Hundsdorfer	$\mathcal{D}_1$ Hundsdorfer
IMEX DeC equi	A(α)-stability $\alpha \approx 35^\circ$ Order 2 strictest stab	Always unstable	Bounded areas increasing with order
IMEX DeC GLB	$\uparrow$	Always unstable	Bounded areas increasing with order
IMEX ADER equi	$\uparrow$	Order 2 stable	Unlimited areas almost A-stable bounded for orders 5 and 8
IMEX ADER GLB	$\uparrow$	Order 2-4 stable	Unlimited areas almost A-stable

Table of contents

Advection – diffusion problems

$$\partial_t u + a\partial_x u - d\partial_{xx} u = 0 \qquad a, d \geq 0$$

Discretization

- Explicit advection term $\frac{a\Delta t}{\Delta x} Du \approx \Delta t a \partial_x u$
- Implicit diffusion term $\frac{d\Delta t}{\Delta x^2} D_2 u \approx \Delta t d \partial_{xx} u$

Advection – diffusion problems

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  - D upwind FD
  - D_2 central FD
- Von Neumann stability analysis

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  - D_2 central FD
- Von Neumann stability analysis
- Many parameters
  - Δt
  - Δx
  - a
  - d
  - wave number k

Advection – diffusion problems

$$\partial_t u + a\partial_x u - d\partial_{xx} u = 0 \qquad a, d \geq 0$$

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  for all $k \in \mathbb{N}$
- Numerically $k = 1, \dots, 1000$

Advection – diffusion problems

$$\partial_t u + a\partial_x u - d\partial_{xx} u = 0 \qquad a, d \geq 0$$

Discretization

- Explicit advection term $\frac{a\Delta t}{\Delta x} D u \approx \Delta t a \partial_x u$
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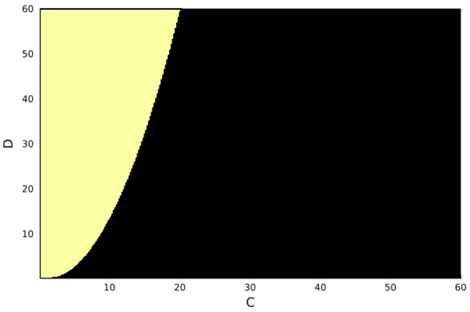

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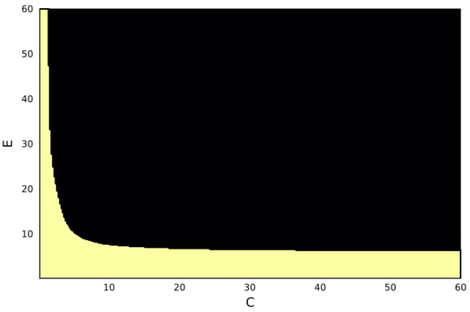
- $C = \frac{a\Delta t}{\Delta x}$
- $E = \frac{C^2}{D} = \frac{a^2\Delta t^2 \Delta x^2}{d\Delta t \Delta x^2} = \frac{a^2 \Delta t}{d}$
- $|G| \leq 1 \, \forall k$



$C-E$ Stability Areas for advection–diffusion

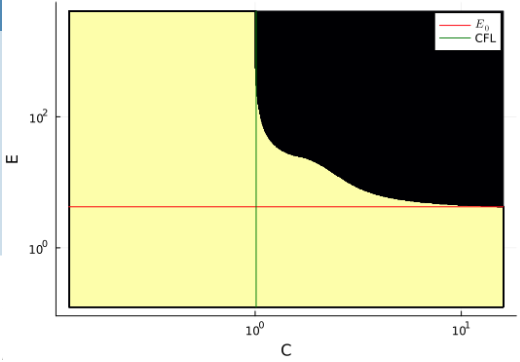
Stability region description (often)

- If $C = \frac{a\Delta t}{\Delta x} \leq C_0 \implies$ Stable
- If $E \leq E_0 \implies$ Stable

$$E = \frac{a^2 \Delta t}{d} \leq E_0 \iff \Delta t \leq \frac{E_0 d}{a^2} =: \tau_0 {}^{a}$$

- Independent on Δx

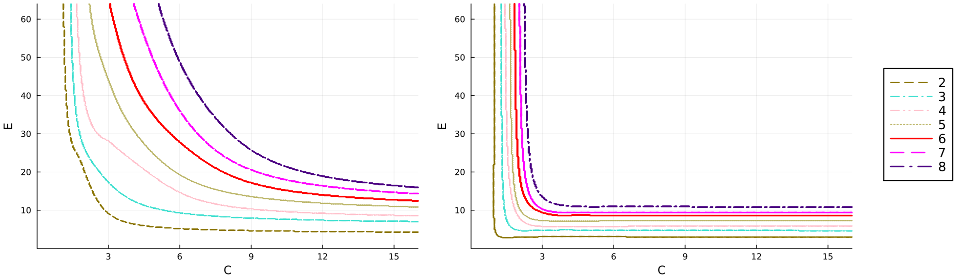
[a] M. Tan, J. Cheng, and C.-W. Shu. Stability of high order finite difference schemes with implicit-explicit time-marching for convection-diffusion and convection-dispersion equations. International Journal of Numerical Analysis and Modeling, 18(3):362-383, 2021.



$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

- Advection $Du_j = \frac{u_j - u_{j-1}}{\Delta x}$ first order
- Diffusion $D_2 u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2}$ second order
- **Time orders** from 2 to 8

Gauss–Lobatto



IMEX DeC IMEX ADER

Figure: Stability areas for orders 2 to 8 with Gauss-Lobatto nodes.

$C-E$ stability plots for IMEX DeC/ADER on advection–diffusion

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Equispaced

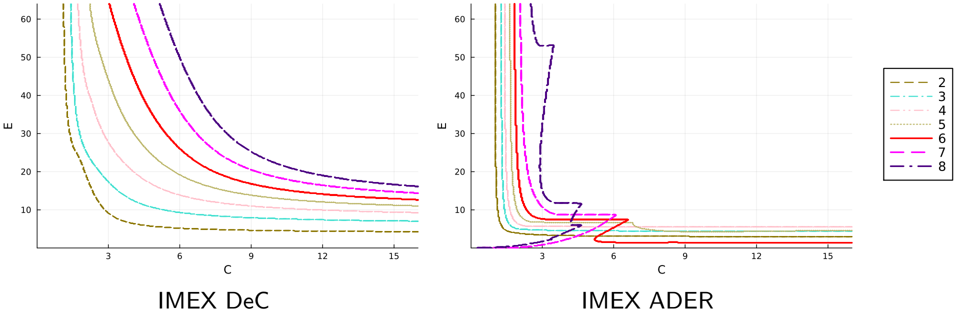
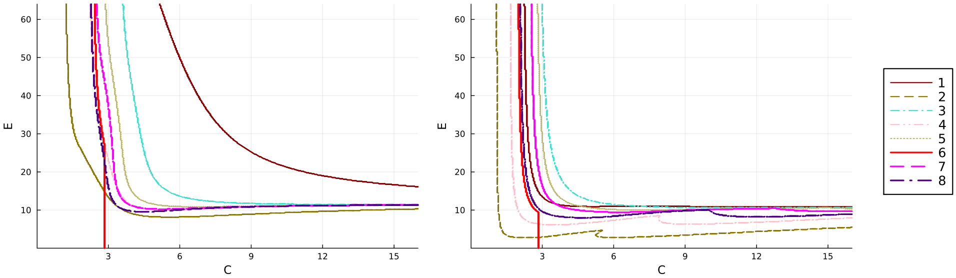


Figure: Stability areas for orders 2 to 8 with equispaced nodes.

$C-E$ stability plots for IMEX DeC/ADER on advection–diffusion

- **Advection operators** order from 1 to 8
- Diffusion $D_2 u_j = \frac{u_{j-1}-2u_j+u_{j+1}}{\Delta x^2}$ second order
- Time order 8

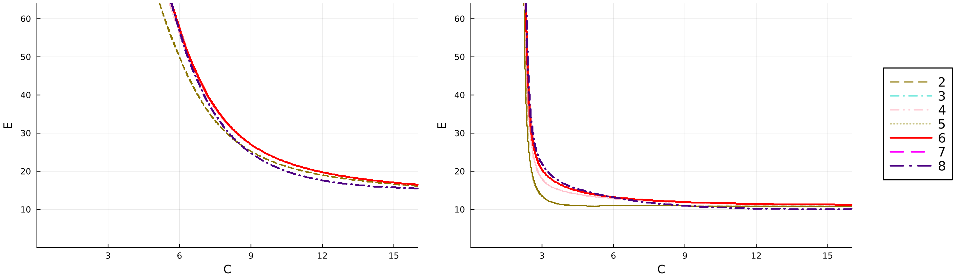


IMEX DeC Equispaced IMEX ADER Gauss–Lobatto

Figure: Stability areas for orders 1 to 8 of the advection operator

$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

- Advection $Du_j = \frac{u_j - u_{j-1}}{\Delta x}$ first order
- **Diffusion operators** central order in $[2, 4, 6, 8]$
- Time order 8



IMEX DeC Equispaced IMEX ADER Gauss–Lobatto

Figure: Stability areas for orders 2 to 8 of the diffusion operator

$C-E$ stability plots for IMEX DeC/ADER on advection–diffusion

- **Advection operator** order k
- **Diffusion operator** order k
- **Time order** k from 2 to 8

Gauss–Lobatto

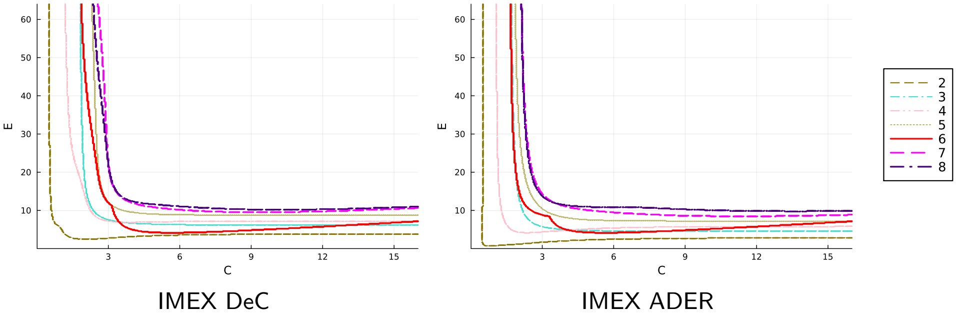


Figure: Stability areas for orders 2 to 8 with Gauss-Lobatto nodes.

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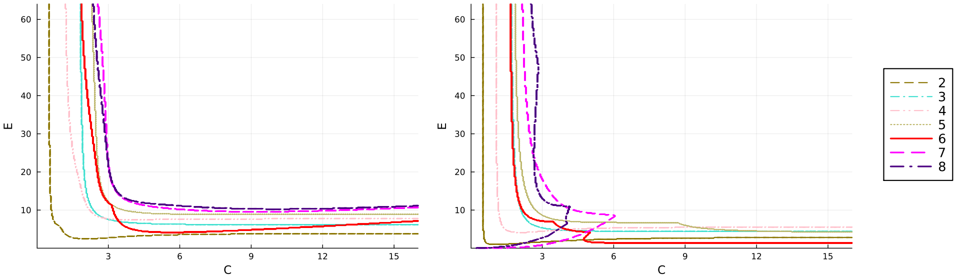
Equispaced



IMEX DeC IMEX ADER

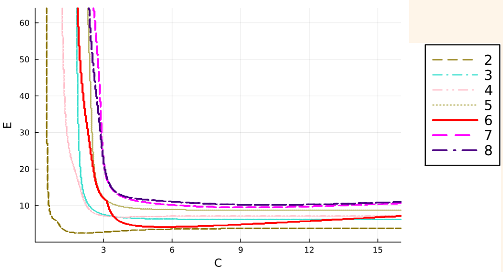
Figure: Stability areas for orders 2 to 8 with equispaced nodes.

C-E stability optimal values

Approximated border values C_0 (up to 2 decimals) and E_0 (up to 1 decimal) for Gauss–Lobatto methods

Order	DeC		ADER	
	C_0	E_0	C_0	E_0
2	0.50	2.5	0.50	0.7
3	1.63	6.1	1.63	4.5
4	1.04	6.9	1.04	4.2
5	1.74	8.8	1.74	7.2
6	1.60	4.1	1.60	4.1
7	1.94	9.5	1.94	8.5
8	2.00	10.2	2.00	9.8

IMEX DeC Gauss–Lobatto



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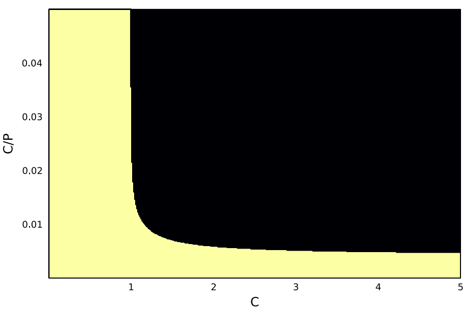

$C - E$ Stability Areas for advection–dispersion

IMEX DeC GLB 2
Advection order 1
Dispersion order 3

Stability region description

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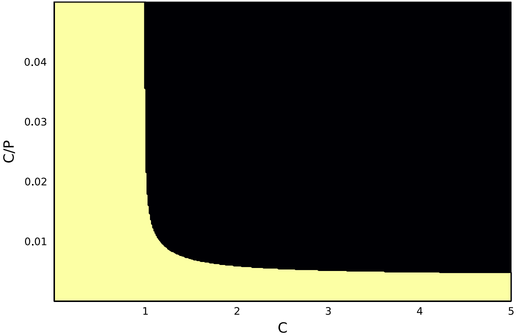
$$E = \frac{a\Delta x^2}{b} \leq E_0 \iff \Delta x \leq \sqrt{\frac{E_0 b}{a}} =: \Delta_{x,0}$$

  - Independent on Δt



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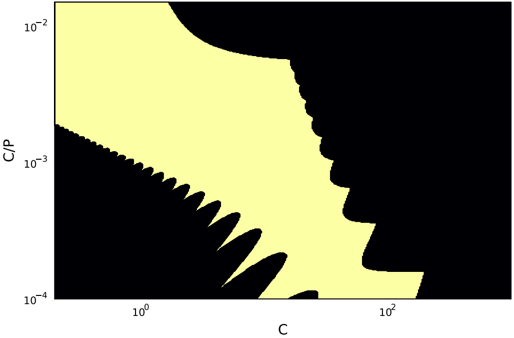
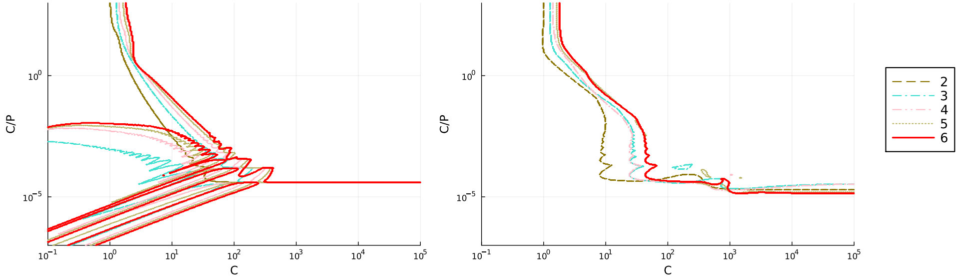


$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

- Advection $Du_j = \frac{u_j - u_{j-1}}{\Delta x}$ first order
- Dispersion $D_3 u_j = \frac{1}{4h^3}\left(-u_{j-2} - u_{j-1} + 10u_j - 14u_{j+1} + 7u_{j+2} - u_{j+3}\right).$ third order
- **Time orders** from 2 to 6

Gauss–Lobatto



IMEX DeC   IMEX ADER

Stability areas for orders 2 to 6 with Gauss-Lobatto nodes.

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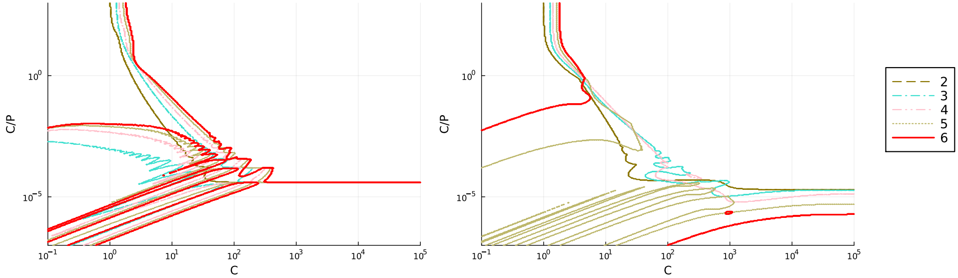

IMEX DeC IMEX ADER

Stability areas for orders 2 to 6 with equispaced nodes.

$C - E$ stability plots for IMEX DeC/ADER on advection–dispersion

- **Advection operator** order k
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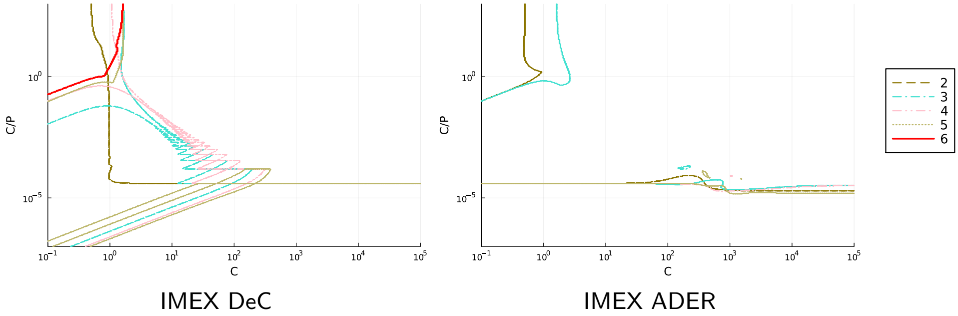

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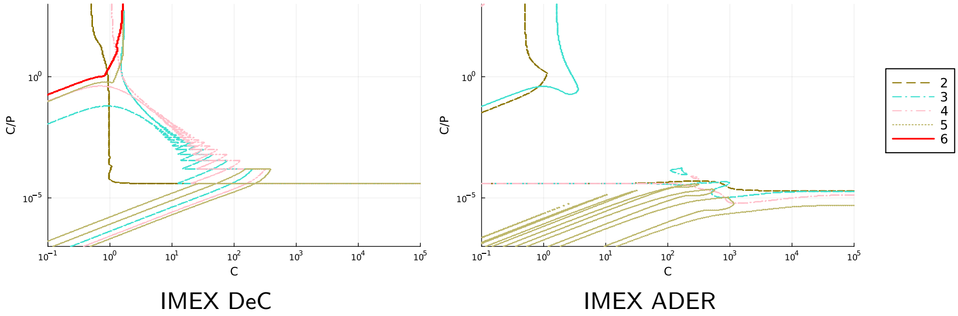

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Other projects with DeC/ADER

- Positivity preserving (Modified Patankar) (Philipp Öffner at 12:00 today)
- Entropy Preserving (Relaxation)
- Efficient version (less stages)
- DOOM a posteriori limiter for ADER-DG in space/time

THANK YOU!

davidetorlo.it

Preprint:
Petri, L., Öffner, P., Torlo, D.. Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations (2024)