High Order Well–Balanced Discrete Kinetic Model for Shallow Water Equations

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joint work with Mario Ricchiuto and Rèmi Abgrall

Outline

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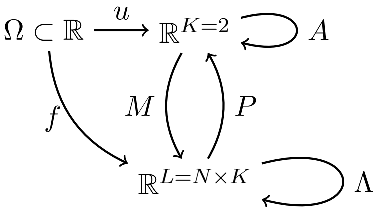
Modify the kinetic relaxation models by D. Aregba-Driollet and R. Natalini[1]
Hyperbolic limit equation is



(1) $$u_t + \partial_x A(u) = 0, \quad u : \Omega \to \mathbb{R}^K$$

(2)

Relaxation system

(3) $$f^\varepsilon = (f_1, f_2, \dots, f_N) = (h_1, q_1, h_2, q_2, \dots, h_N, q_N)$$

(4) $$f^\varepsilon_t + \Lambda \partial_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right), \quad f^\varepsilon : \Omega \to \mathbb{R}^L$$

$$Pf^\varepsilon \to u, \quad P(M(u)) = u, \quad P\Lambda M(u) = A(u) \qquad .$$

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Kinetic and Shallow water equations

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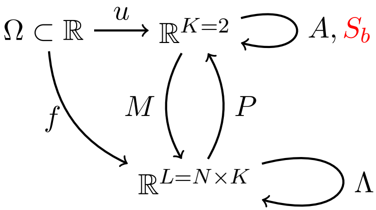

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(2) $$\begin{cases} h_t + q_x = 0 \\ q_t + (q^2/h + \frac{g}{2}h^2)_x + ghb_x = 0 \end{cases}$$

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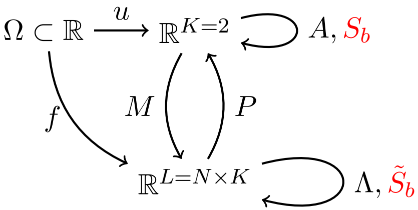

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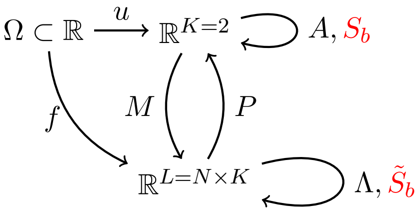

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Kinetic model – DRM

$$f^{\varepsilon} = (f_1, f_2)^T = (h_1, q_1, h_2, q_2)^T$$

Diagonal relaxation method (DRM)

- $K = 2$
- $N = D + 1 = 2$
- $L = N \times K = 2 \times 2$
- $P = (I_K, \dots, I_K) = (I_2, I_2)$
- $\Lambda = \begin{pmatrix} -\lambda I_2 & \\ & \lambda I_2 \end{pmatrix}$
- $M_1(u) = \frac{u\lambda - A(u)}{2\lambda}$
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$$u^\varepsilon := Pf^\varepsilon = f_1 + f_2$$

$$h^\varepsilon = h_1 + h_2, \quad q^\varepsilon = q_1 + q_2$$

$$f^\varepsilon_t + \Lambda \partial_x f^\varepsilon + \tilde{S}_b(f^\varepsilon) = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right)$$

$$\partial_t \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix} + \partial_x \begin{pmatrix} -\lambda h_1 \\ -\lambda q_1 \\ \lambda h_2 \\ \lambda q_2 \end{pmatrix} - \begin{pmatrix} 0 \\ g(h_1 + \frac{b}{2})\partial_x b \\ 0 \\ g(h_2 + \frac{b}{2})\partial_x b \end{pmatrix} =$$

$$\frac{1}{2\varepsilon} \begin{pmatrix} -h_1 + h_2 - \frac{q^\varepsilon}{\lambda} \\ -q_1 + q_2 - \frac{(q^\varepsilon)^2/(h^\varepsilon) + g((h^\varepsilon)^2 - b^2)/2}{\lambda} \\ h_1 - h_2 + \frac{q^\varepsilon}{\lambda} \\ q_1 - q_2 + \frac{(q^\varepsilon)^2/(h^\varepsilon) + g((h^\varepsilon)^2 - b^2)/2}{\lambda} \end{pmatrix}.$$

Chapman–Enskog

Relaxation system

$$f_t^\varepsilon + \Lambda \partial_x f^\varepsilon + \tilde{S}_b(f^\varepsilon) = \frac{M(Pf^\varepsilon) - f^\varepsilon}{\varepsilon},$$

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$$\partial_t u^\varepsilon + \partial_x A(u^\varepsilon) + S_b(u^\varepsilon) = \varepsilon\Xi + \mathcal{O}(\varepsilon^2), \quad u^\varepsilon = Pf^\varepsilon,$$

where $\Xi := \partial_x(B(u^\varepsilon)\partial_x u^\varepsilon) + \partial_x(-A'(u^\varepsilon)S_b(u^\varepsilon) + S_b(A(u^\varepsilon))),$

with $B(u) := P\Lambda^2 M'(u) - A'(u)^2 \in \mathbb{R}^{S\times S}.$

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Whitham's subcharacteristic condition $B \geq 0 \quad \Longrightarrow \quad$ Diffusive

Outline

Residual Distribution[2]

- High order
- FE based
- Compact stencil
- No need of conservative variables
- Can recast some other FV, FE schemes[2]

Finite Element Setting

$$\partial_t f + \nabla_x \cdot A(f) = S(f)$$

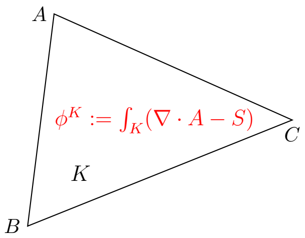
$$V_h = \{f \in L^2(\Omega_h, \mathbb{R}^L) \cap \mathcal{C}^0(\Omega_h), \\ f|_K \in \mathbb{P}^p, \forall K \in \Omega_h\}$$

$$f(x) = \sum_{\sigma \in D_h} f_\sigma \varphi_\sigma(x) \\ = \sum_{K \in \Omega_h} \sum_{\sigma \in K} f_\sigma \varphi_\sigma(x)|_K$$

[2] R. Abgrall. Some remarks about conservation for residual distribution schemes. Computational Methods in Applied Mathematics, 2018. DOI: https://doi.org/10.1515/cmam-2017-0056.

Residual Distribution - Spatial Discretization

1. Define $\forall K \in \Omega_h$ a fluctuation term (total residual) $\phi^K = \int_K \nabla \cdot A(f) - S(f) dx$



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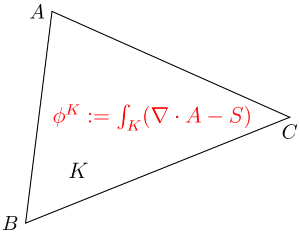
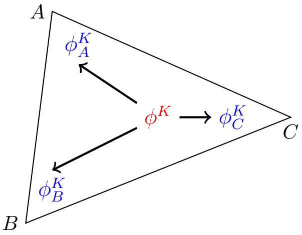
Choice of Residuals

Basic algorithm (Galerkin), numerical fluxes (Rusanov), linear stabilization terms (SUPG, jump derivative penalty), non linear stabilization (PSI).





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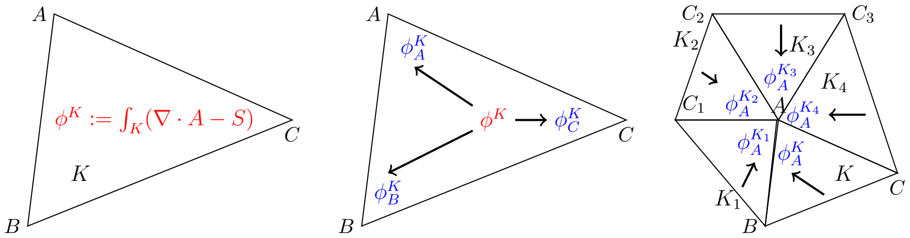
3. The resulting scheme is $\partial_t f_\sigma + \sum_{K|\sigma \in K} \phi^K_\sigma = 0, \quad \forall \sigma \in D_h.$



Outline

IMEX discretization - Kinetic model

Stiff source term $\Rightarrow$ unstable for $\varepsilon \ll \Delta t \Rightarrow$ IMEX approach:
IMplicit for stiff source term, EXplicit for advection term and bathymetry source

$$\frac{f^{n+1,\varepsilon} - f^{n,\varepsilon}}{\Delta t} + \Lambda \partial_x f^{n,\varepsilon} + \tilde{S}_b(f^{n,\varepsilon}) = \frac{1}{\varepsilon}\left(M(Pf^{n+1,\varepsilon}) - f^{n+1,\varepsilon}\right). \tag{5}$$

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How to treat non-linear implicit functions?
Recall: $PM(u)=u$ and $Pf^\varepsilon=u^\varepsilon$, so

$$\frac{u^{n+1,\varepsilon}-u^{n,\varepsilon}}{\Delta t}+P\Lambda\partial_x f^{n,\varepsilon}+S_b(u^{n,\varepsilon})=0. \tag{6}$$

Find $u^{n+1,\varepsilon}=Pf^{n+1,\varepsilon}$ and substitute it in the Maxwellian in (5).

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- IMEX formulation is first order accurate =: $\mathcal{L}^1$
- IMEX formulation is asymptotic preserving (AP) (as $\varepsilon \to 0$ we recast SW)

Deferred Correction[3]

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$$f^{0,(k)} := f(t^n), \quad k = 0, \dots, K,$$

$$f^{m,(0)} := f(t^n), \quad m = 1, \dots, M$$

$\mathcal{L}^1(f^{(k)}) = \mathcal{L}^1(f^{(k-1)}) - \mathcal{L}^2(f^{(k-1)})$ with $k = 1, \dots, K$.

$\mathcal{L}^1(f) = 0$

- IMEX
- First order accurate
- Mass lumping
- Computationally explicit

$\mathcal{L}^2(f) = 0$

- Order $M+1$
- Quadrature in timestep
- Nonlinearly implicit
- Implicit Runge–Kutta

DeC Theorem

- $\mathcal{L}^1$ coercive
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz

DeC converges and $\min(K, M+1)$ is the order of accuracy.

[3] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

Outline

Other properties

- Well balancedness: lake at rest steady state preservation
  - Match of the discretizations of the source term and the flux when $v=0$ and
    $\eta(x) = h^\varepsilon(x) + b(x) \equiv \eta_0$
  - $\phi_\sigma^K = \int_K g\varphi^\sigma \partial_x \frac{(h^\varepsilon)^2 - b^2}{2} dx + \int_E g\varphi^\sigma (h^\varepsilon + b)\partial_x b \, dx = 0$

$$\int_K g\varphi^\sigma \partial_x \varphi^i(x) \underbrace{\frac{h^\varepsilon(x_i) - b(x_i)}{2}}_{=\frac{\eta_0}{2} - b(x_i)} \underbrace{(h^\varepsilon(x_i) + b(x_i))}_{=\eta_0} dx = \int_K -g\varphi^\sigma \eta_0 \partial_x \varphi^i(x) b(x_i) dx =$$

$$-\int_E g\varphi^\sigma (h^\varepsilon + b)\partial_x b \, dx.$$

  - Recipe for all sources $\tilde{S}_b$
  - Stabilization techniques depends on η instead of h

Other properties

- Depth non-negativity
  - Wet and dry elements: dry elements $h_1 + h_2 \leq \texttt{tol}_0$, $q_1 = q_2 = 0$
  - Hybrid elements $\Longrightarrow$ Lower the bathymetry to have positive DoFs where $h_1 + h_2 \leq \texttt{tol}_0$ (IC or along the scheme) to preserve the well–balancedness $\eta_K \equiv C$
  - Use of explicit schemes that, subjected to CFL conditions, can preserve the non–negativity, e.g. Rusanov, modified Rusanov, blended PSI+Rusanov. (Not jump stabilization)

Outline

Simulations: Convergence for Subcritical Flow

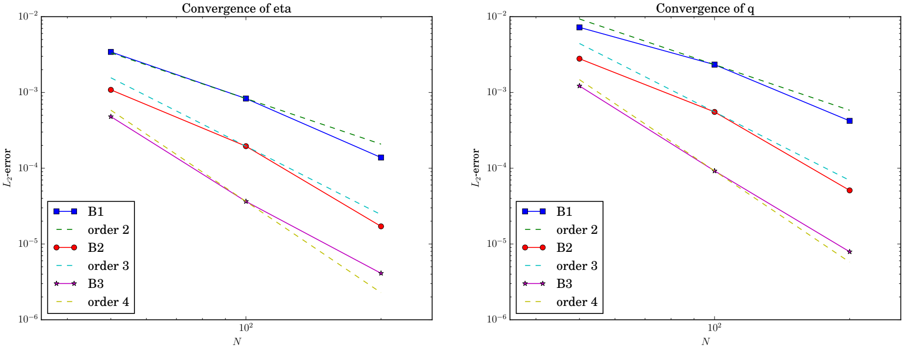

Figure: Subcritical flow: convergence for $\eta^\varepsilon = h^\varepsilon + b$ and $h^\varepsilon v^\varepsilon$

$$b(x) = \begin{cases} 0.2 \exp\left(\frac{((x-10)/5)^2}{1-((x-10)/5)^2}\right), & \text{if } x \in B_5(10), \\ 0, & \text{else.} \end{cases}$$

$$h^\varepsilon(0,x) = 2 - b(x) \qquad q^\varepsilon(0,t) = 4.42$$

$$q^\varepsilon(0,x) = 4.42 \qquad h^\varepsilon(25,t) = 2$$

$$f^\varepsilon(0,x) = M(u^\varepsilon(0,x)) \qquad T = 100$$

$$\lambda = 6.5, \quad \varepsilon = 10^{-14},$$

Simulation: transcritical with shock

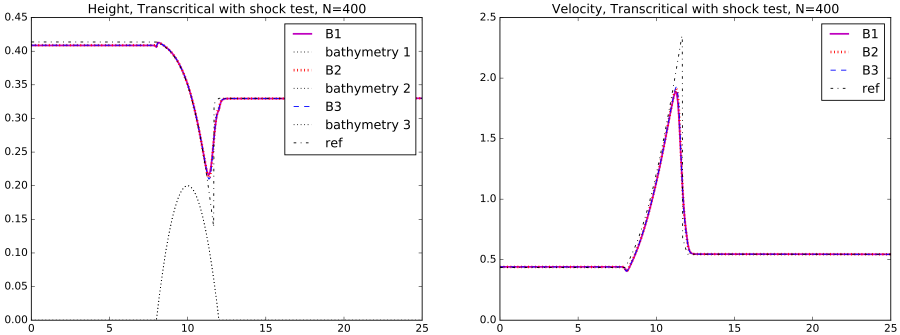

Figure: Transcritical flow with shock test: η^ε and v^ε with $N = 400$

$$b(x) = (0.2 - 0.05(x-10)^2)\mathbb{1}_{\{8<x<12\}}$$
$$\eta^\varepsilon(0,x) = 0.4 - 0.07)\mathbb{1}_{\{x>8\}}$$
$$q^\varepsilon(0,x) = 0.14$$

$$q^\varepsilon(0,t) = 0.18$$
$$h^\varepsilon(25,t) = 0.33$$
$$\lambda = 4, \quad \varepsilon = 10^{-14}.$$

Simulations: lake at rest

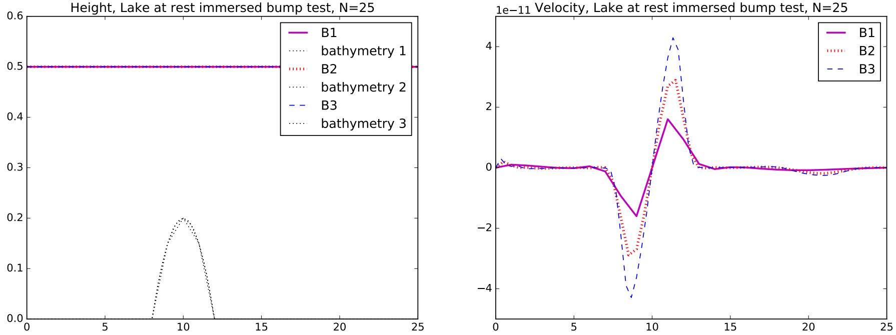

Figure: Lake at rest with immersed bump test: η^ε and v^ϵ with $N = 25$

$$b(x) = (0.2 - 0.05(x-10)^2)\mathbb{1}_{\{8<x<12\}} \qquad q^\varepsilon(0,t) = 0 \qquad q - q^{ex} = \mathcal{O}(N_t \varepsilon)$$

$$\eta^\varepsilon(0,x) = 0.5 \qquad q^\varepsilon(25,t) = 0 \qquad T = 3$$

$$q^\varepsilon(0,x) = 0 \qquad \lambda = 2 \qquad \varepsilon = 10^{-14}$$

Simulations: wet and dry lake at rest

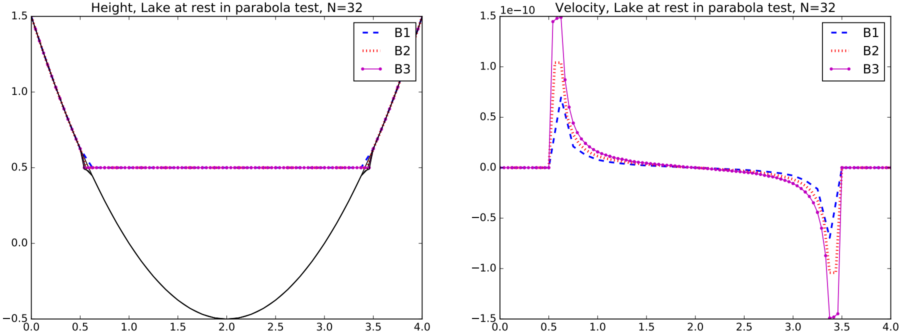

Figure: Lake at rest in parabola test: η^ε and v^ε with $N = 32$

$$b(x) = (x-2)^2 - 0.5$$
$$\eta^\varepsilon(0,x) = \max(0.5, b(x))$$
$$\lambda = 4$$

$$q - q^{ex} = \mathcal{O}\left(N_t \varepsilon\right)$$
$$T = 3$$
$$\varepsilon = 10^{-14}$$

Simulations: Thucker Oscillations

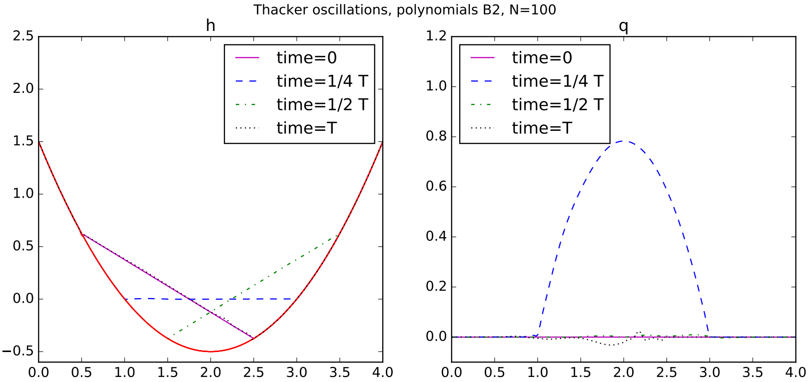

Figure: Thacker oscillations in parabola test: η^ε and $h^\varepsilon v^\varepsilon$ with $N = 100$

$$b(x) = (x-2)^2 - 0.5$$

$$\eta^\varepsilon(0,x) = \max(-0.5x + 0.875, b(x))$$

$$\lambda = 6.5$$

$$\text{period} = 2.0606$$

$$T = 5 \cdot 2.0606$$

$$\varepsilon = 10^{-14}$$

Outline

Conclusion and perspective

Conclusions

- IMEX
- High order
- Residual Distribution
- Deferred Correction
- Well–balanced
- Wet/dry
- Nonnegative water height

Perspective

- MOOD
- Entropy stability
- Multi dimension

IMEX DeC RD – Bibliography

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Thank you for the attention!