Continuous Galerkin high order well-balanced discrete kinetic model for shallow water equations

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joint work with Mario Ricchiuto and Rémi Abgrall

Outline

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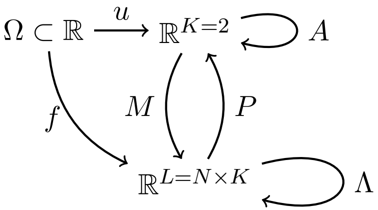
Kinetic and Shallow water equations

Modify the kinetic relaxation models by D. Aregba-Driollet and R. Natalini[1]
Hyperbolic limit equation is

(1) $\quad u_t + \partial_x A(u) \quad = 0, \quad u : \Omega \to \mathbb{R}^K$

(2)



Relaxation system

(3) $\quad f^\varepsilon = (f_1, f_2, \dots, f_N) = (h_1, q_1, h_2, q_2, \dots, h_N, q_N)$

(4) $\quad f^\varepsilon_t + \Lambda \partial_x f^\varepsilon \quad = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right), \quad f^\varepsilon : \Omega \to \mathbb{R}^L$

$$Pf^\varepsilon \to u, \quad P(M(u)) = u, \quad P\Lambda M(u) = A(u) \quad .$$

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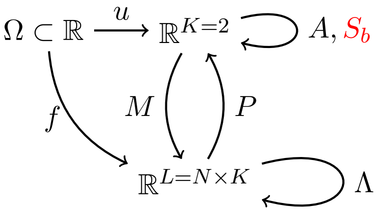

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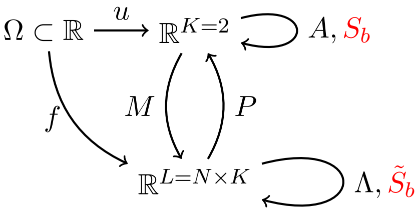

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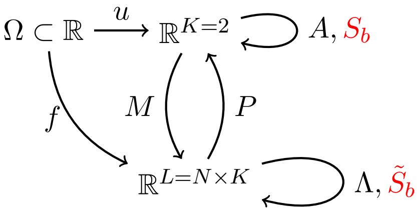

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Kinetic model – DRM

$$f^\varepsilon = (f_1, f_2)^T = (h_1, q_1, h_2, q_2)^T$$

Diagonal relaxation method (DRM)

- $K = 2$
- $N = D + 1 = 2$
- $L = N \times K = 2 \times 2$
- $P = (I_K, \dots, I_K) = (I_2, I_2)$
- $\Lambda = \begin{pmatrix} -\lambda I_2 & \\ & \lambda I_2 \end{pmatrix}$
- $M_1(u) = \frac{u\lambda - A(u)}{2\lambda}$
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$$u^\varepsilon := Pf^\varepsilon = f_1 + f_2$$

$$h^\varepsilon = h_1 + h_2, \quad q^\varepsilon = q_1 + q_2$$

$$f_t^\varepsilon + \Lambda \partial_x f^\varepsilon + \tilde{S}_b(f^\varepsilon) = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right)$$

$$\partial_t \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix} + \partial_x \begin{pmatrix} -\lambda h_1 \\ -\lambda q_1 \\ \lambda h_2 \\ \lambda q_2 \end{pmatrix} - \begin{pmatrix} 0 \\ g(h_1 + \frac{b}{2})\partial_x b \\ 0 \\ g(h_2 + \frac{b}{2})\partial_x b \end{pmatrix} =$$

$$\frac{1}{2\varepsilon} \begin{pmatrix} -2h_1 + h^\varepsilon - \frac{q^\varepsilon}{\lambda} \\ -2q_1 + q^\varepsilon - \frac{(q^\varepsilon)^2/(h^\varepsilon) + g((h^\varepsilon)^2 - b^2)/2}{\lambda} \\ -2h_2 + h^\varepsilon + \frac{q^\varepsilon}{\lambda} \\ -2q_2 + q^\varepsilon + \frac{(q^\varepsilon)^2/(h^\varepsilon) + g((h^\varepsilon)^2 - b^2)/2}{\lambda} \end{pmatrix}.$$

Chapman–Enskog

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$$\partial_t u^\varepsilon + \partial_x A(u^\varepsilon) + S_b(u^\varepsilon) = \varepsilon\Xi + \mathcal{O}(\varepsilon^2), \quad u^\varepsilon = Pf^\varepsilon,$$

where $\Xi := \underbrace{\partial_x(B(u^\varepsilon)\partial_x u^\varepsilon)}_{\text{diffusion}} + \underbrace{\partial_x(-A'(u^\varepsilon)S_b(u^\varepsilon) + S_b(A(u^\varepsilon)))}_{?},$

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Whitham's subcharacteristic condition $B \geq 0 \quad \Longrightarrow \quad$ Diffusive

Chapman Enskog of Global Flux

- Liu et al., SIAM Journal on Scientific Computing, 42 (2020)

(5) $$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x w = 0 \\ \partial_t w + a^2 \partial_x q = \frac{G(h,q)-w}{\varepsilon}, \end{cases}$$

$$\begin{aligned} G(h,q) &:= F(h,q) + \Sigma(h,q) \\ \Sigma(h,q) &:= \int^x S(h,q)dx \\ F(h,q) &:= \tfrac{q^2}{h} + \tfrac{g}{2}h^2 \\ S(h,q) &:= ghb_x \end{aligned}$$

Chapman–Enskog

$$\partial_t q + \partial_x G = \varepsilon\, \underbrace{\partial_x \left[\left(a^2 - \frac{\partial G}{\partial h} - \left(\frac{\partial G}{\partial q}\right)^2 \right) \partial_x q - \frac{\partial G}{\partial q}\frac{\partial G}{\partial h} \partial_x h \right]}_{\text{Diffusion?}},$$

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Outline

Residual Distribution[2]

- High order
- FE based
- Compact stencil
- No need of conservative variables
- Can recast some other FV, FE, GF, DG schemes[2]

Finite Element Setting

$$\partial_t f + \nabla_x \cdot A(f) = S(f)$$

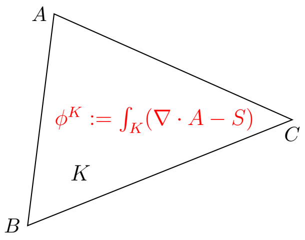
$$V_h = \{ f \in L^2(\Omega_h, \mathbb{R}^L) \cap \mathcal{C}^0(\Omega_h), \\ f|_K \in \mathbb{P}^p, \, \forall K \in \Omega_h \}$$

$$f(x) = \sum_{\sigma \in D_h} f_\sigma \varphi_\sigma(x) \\ = \sum_{K \in \Omega_h} \sum_{\sigma \in K} f_\sigma \varphi_\sigma(x)|_K$$

[2] R. Abgrall. Some remarks about conservation for residual distribution schemes. Computational Methods in Applied Mathematics, 2018. DOI: https://doi.org/10.1515/cmam-2017-0056.

Residual Distribution - Spatial Discretization

1. Define $\forall K \in \Omega_h$ a fluctuation term (total residual) $\phi^K = \int_K \nabla \cdot A(f) - S(f) dx$



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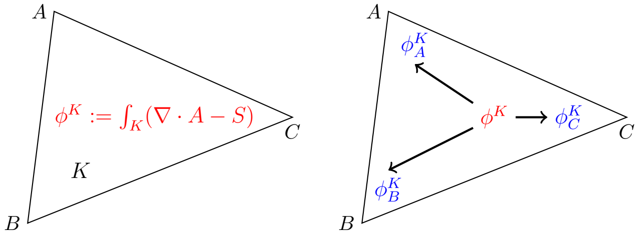
2. Define nodal residuals ϕ^K_σ $\forall \sigma \in K : \phi^K = \sum_{\sigma \in K} \phi^K_\sigma, \quad \forall K \in \Omega_h$.

Choice of Residuals

Basic algorithm (Galerkin), numerical fluxes (Rusanov), linear stabilization terms (SUPG, jump derivative penalty), non linear stabilization (PSI).



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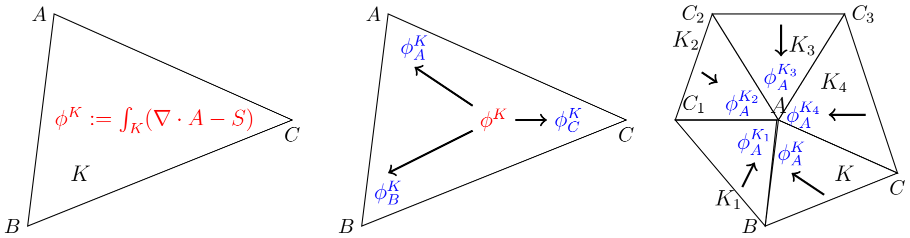
3. The resulting scheme is $\partial_t f_\sigma + \sum_{K|\sigma \in K} \phi^K_\sigma = 0, \quad \forall \sigma \in D_h.$



Outline

IMEX discretization - Kinetic model

Stiff source term $\Rightarrow$ unstable for $\varepsilon \ll \Delta t \Rightarrow$ IMEX approach:
IMplicit for stiff source term, EXplicit for advection term and bathymetry source

$$\frac{f^{n+1,\varepsilon} - f^{n,\varepsilon}}{\Delta t} + \Lambda \partial_x f^{n,\varepsilon} + \tilde{S}_b(f^{n,\varepsilon}) = \frac{1}{\varepsilon}\left(M(Pf^{n+1,\varepsilon}) - f^{n+1,\varepsilon}\right). \tag{6}$$

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How to treat non-linear implicit functions?
Recall: $PM(u) = u$ and $Pf^{\varepsilon} = u^{\varepsilon}$, so

(7) $$\frac{u^{n+1,\varepsilon} - u^{n,\varepsilon}}{\Delta t} + P\Lambda \partial_x f^{n,\varepsilon} + S_b(u^{n,\varepsilon}) = 0.$$

Find $u^{n+1,\varepsilon} = Pf^{n+1,\varepsilon}$ and substitute it in the Maxwellian in (6).

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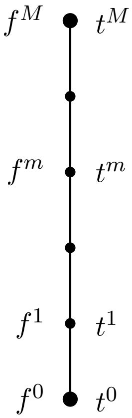
- IMEX formulation is first order accurate =: $\mathcal{L}^1$
- IMEX formulation is asymptotic preserving (AP) (as $\varepsilon \to 0$ we recast SW)

Deferred Correction[3]



[3] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

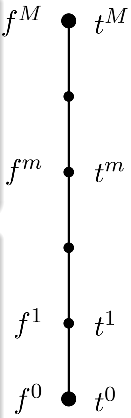
Deferred Correction[3]

$\mathcal{L}^1(f)=0$

- IMEX
- First order accurate
- Mass lumping
- Computationally explicit

$\mathcal{L}^2(f)=0$

- Order $M+1$
- Quadrature in timestep
- Nonlinearly implicit
- Implicit Runge–Kutta



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$$f^{0,(k)} := f(t^n), \quad k = 0, \dots, K,$$
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DeC Theorem

- $\mathcal{L}^1$ coercive
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz

DeC order accuracy $\min(K, M+1)$.

AP Theorem

$\mathcal{L}^1$ AP $\implies$ DeC AP

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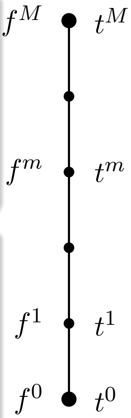

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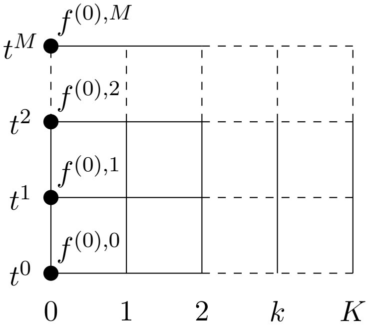

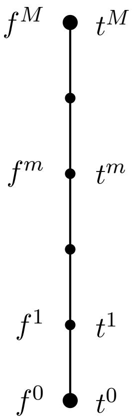

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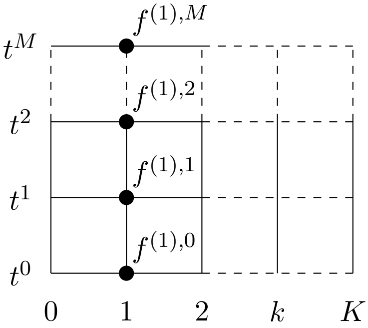

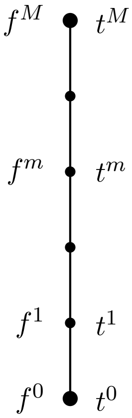

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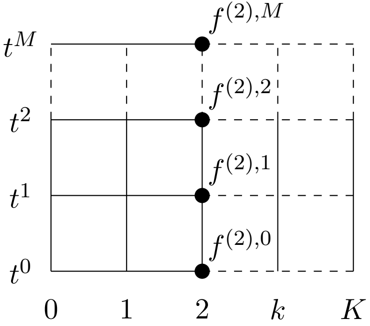

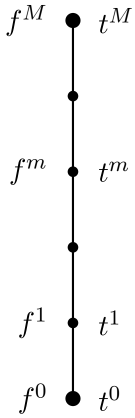

DeC Theorem

- $\mathcal{L}^1$ coercive
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz

DeC order accuracy $\min(K, M+1)$.

AP Theorem[4]

$\mathcal{L}^1$ AP $\Longrightarrow$ DeC AP

[4] R. Abgrall, and D.T.. SIAM Journal on Scientific Computing, 42(3):B816–B845, 2020.

Deferred Correction[4]

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$$f^{0,(k)} := f(t^n), \quad k = 0, \dots, K,$$
$$f^{m,(0)} := f(t^n), \quad m = 1, \dots, M$$
$$\mathcal{L}^1(\underline{f}^{(k)}) = \mathcal{L}^1(\underline{f}^{(k-1)}) - \mathcal{L}^2(\underline{f}^{(k-1)}), k \leq K.$$

- $\mathcal{L}^1(\underline{f}) = 0$, first order accuracy, easily invertible.
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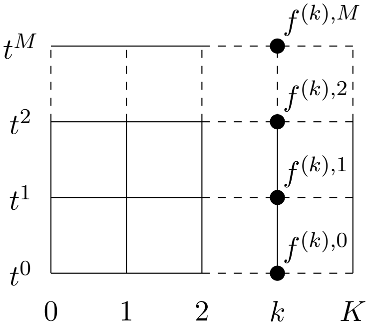

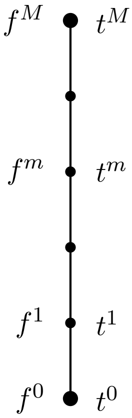

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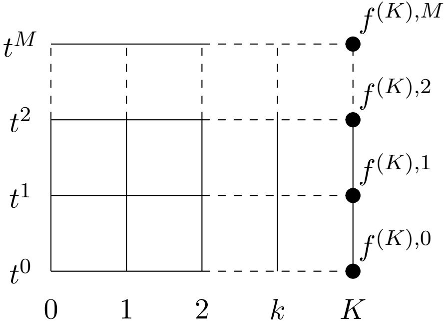

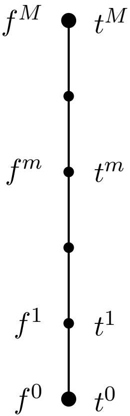

DeC Theorem

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$\mathcal{L}^1$ AP $\Longrightarrow$ DeC AP

[4] R. Abgrall, and D.T.. SIAM Journal on Scientific Computing, 42(3):B816–B845, 2020.

Outline

Other properties

- Well balancedness: lake at rest steady state preservation
  - Match of the discretizations of the source term and the flux when $v = 0$ and
    $\eta(x) = h^\varepsilon(x) + b(x) \equiv \eta_0$
  - $\phi^K_\sigma = \int_K g\varphi^\sigma \partial_x \frac{(h^\varepsilon)^2 - b^2}{2} dx + \int_E g\varphi^\sigma (h^\varepsilon + b)\partial_x b \, dx = 0$

$$\int_K g\varphi^\sigma \partial_x \varphi^i(x) \underbrace{\frac{h^\varepsilon(x_i) - b(x_i)}{2}}_{=\frac{\eta_0}{2} - b(x_i)} \underbrace{(h^\varepsilon(x_i) + b(x_i))}_{=\eta_0} dx = \int_K -g\varphi^\sigma \eta_0 \partial_x \varphi^i(x) b(x_i) dx =$$

$$- \int_E g\varphi^\sigma (h^\varepsilon + b)\partial_x b \, dx.$$

  - Recipe for all sources $\tilde{S}_b$
  - Stabilization techniques depends on η instead of h
- Depth non-negativity: mark dry cells, use positive schemes (Rusanov, modified Rusanov, PSI)

Outline

Simulations: Subcritical Flow

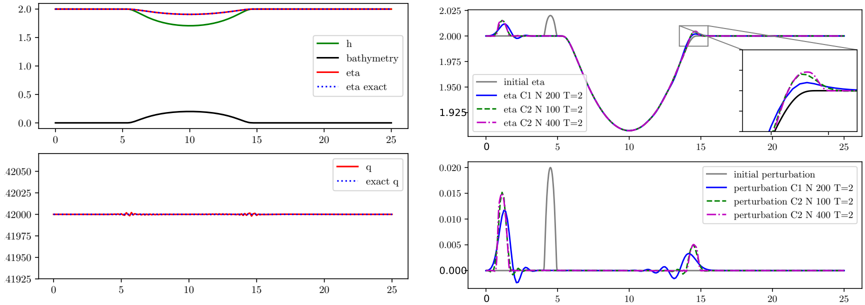

Figure: Subcritical flow: simulation $\mathbb{C}^2$, $N = 100$ (left) perturbation propagating (right)

$$b(x) = \begin{cases} 0.2\exp\left(\frac{((x-10)/5)^2}{1-((x-10)/5)^2}\right), & \text{if } x \in B_5(10), \\ 0, & \text{else.} \end{cases}$$

$$h^\varepsilon(0,x) = 2 - b(x) \qquad q^\varepsilon(0,t) = 4.42$$

$$q^\varepsilon(0,x) = 4.42 \qquad h^\varepsilon(25,t) = 2$$

$$f^\varepsilon(0,x) = M(u^\varepsilon(0,x)) \qquad T = 100$$

$\lambda = 6.5, \quad \varepsilon = 10^{-14},$

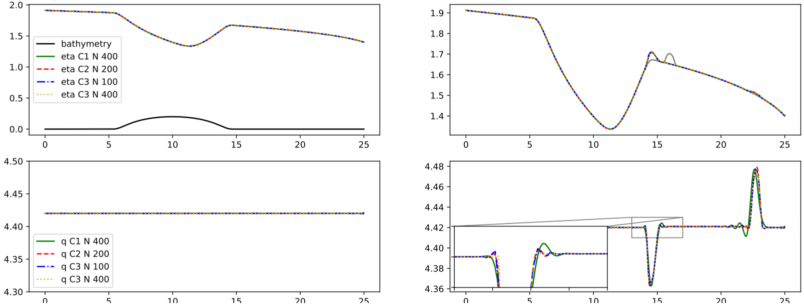

Figure: Subcritical flow: simulation with friction $\mathbb{C}^2$, $N = 100$ (left) perturbation propagating (right)

$$b(x) = \begin{cases} 0.2\exp\left(\frac{((x-10)/5)^2}{1-((x-10)/5)^2}\right), & \text{if } x \in B_5(10), \\ 0, & \text{else.} \end{cases}$$

$$h^\varepsilon(0,x) = 2 - b(x) \qquad q^\varepsilon(0,t) = 4.42$$

$$q^\varepsilon(0,x) = 4.42 \qquad h^\varepsilon(25,t) = 2$$

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Simulations: Subcritical Flow

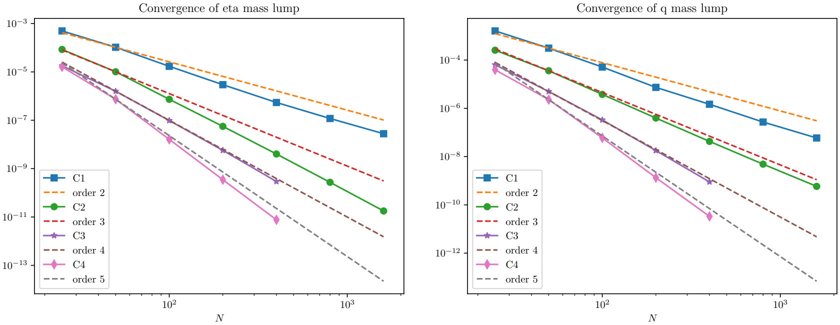

Figure: Subcritical flow: convergence for $\eta^\varepsilon = h^\varepsilon + b$ and $h^\varepsilon v^\varepsilon$

$$b(x) = \begin{cases} 0.2\exp\left(\frac{((x-10)/5)^2}{1-((x-10)/5)^2}\right), & \text{if } x \in B_5(10), \\ 0, & \text{else.} \end{cases} \quad h^\varepsilon(0,x) = 2 - b(x) \quad q^\varepsilon(0,t) = 4.42$$

$$q^\varepsilon(0,x) = 4.42 \quad h^\varepsilon(25,t) = 2$$

$$\lambda = 6.5, \quad \varepsilon = 10^{-14}, \quad f^\varepsilon(0,x) = M(u^\varepsilon(0,x)) \quad T = 100$$

Simulations: lake at rest

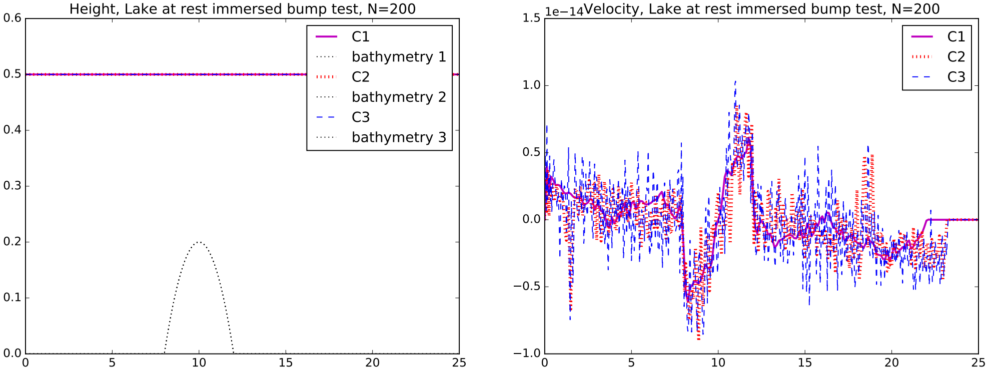

Figure: Lake at rest with immersed bump test: η^ε and v^ϵ with $N = 200$

$$b(x) = (0.2 - 0.05(x-10)^2)\mathbb{1}_{\{8<x<12\}} \qquad q^\varepsilon(0,t) = 0 \qquad q - q^{ex} = \mathcal{O}(N_t \varepsilon)$$

$$\eta^\varepsilon(0,x) = 0.5 \qquad q^\varepsilon(25,t) = 0 \qquad T = 3$$

$$q^\varepsilon(0,x) = 0 \qquad \lambda = 2 \qquad \varepsilon = 10^{-14}$$

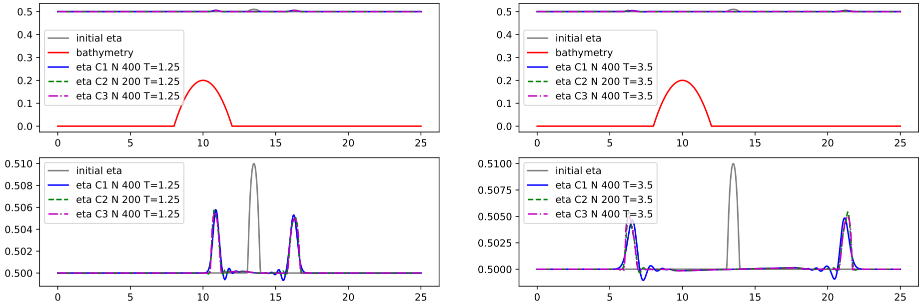

Figure: Lake at rest with immersed bump perturbed: η^ε and v^ϵ with $N = 200$

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$$\eta^\varepsilon(0,x) = 0.5 \qquad q^\varepsilon(25,t) = 0 \qquad T = 3$$

$$q^\varepsilon(0,x) = 0 \qquad \lambda = 2 \qquad \varepsilon = 10^{-14}$$

Simulations: wet and dry lake at rest

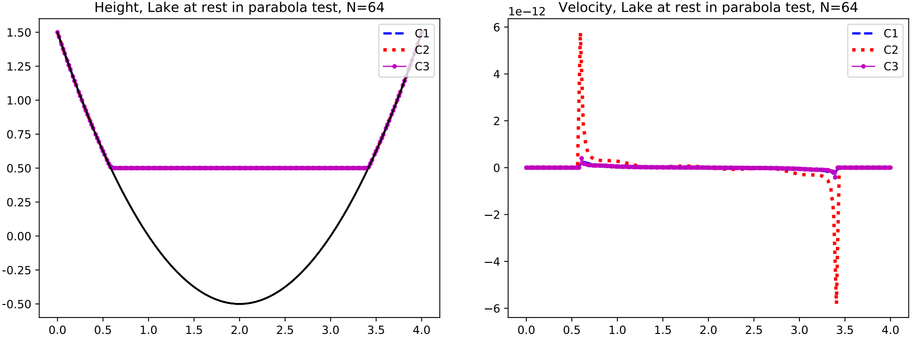

Figure: Lake at rest in parabola test: η^ε and v^ε with $N = 64$

$$b(x) = (x-2)^2 - 0.5$$
$$\eta^\varepsilon(0,x) = \max(0.5, b(x))$$
$$\lambda = 4$$

$$q - q^{ex} = \mathcal{O}(N_t \varepsilon)$$
$$T = 3$$
$$\varepsilon = 10^{-14}$$

Simulations: Thacker's Oscillations

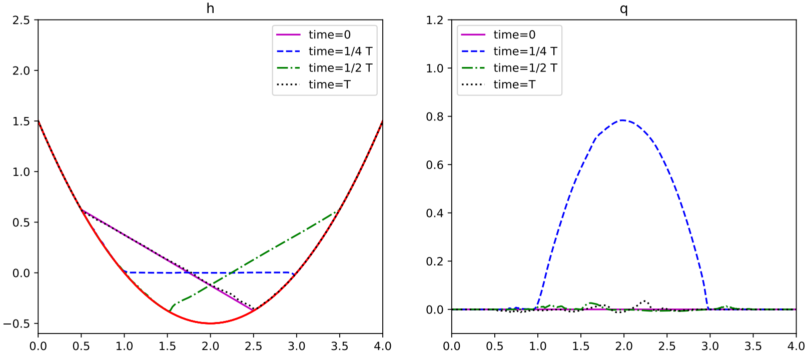

Figure: Thacker oscillations in parabola test: η^ε and $h^\varepsilon v^\varepsilon$ with $\mathbb{C}^1$ and $N = 300$

$$b(x) = (x-2)^2 - 0.5$$

$$\eta^\varepsilon(0,x) = \max(-0.5x + 0.875, b(x))$$

$$\lambda = 6.5$$

period $= 2.0606$

$$T = 5 \cdot 2.0606$$

$$\varepsilon = 10^{-14}$$

Simulations: Thacker's Oscillations

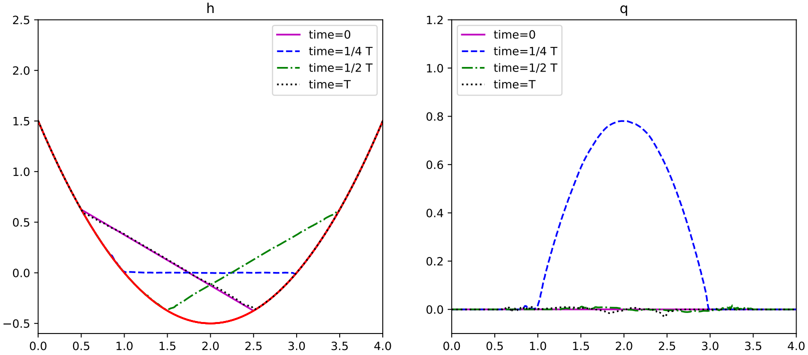

Figure: Thacker oscillations in parabola test: η^ε and $h^\varepsilon v^\varepsilon$ with $\mathbb{C}^2$ and $N = 300$

$$b(x) = (x-2)^2 - 0.5$$

$$\eta^\varepsilon(0,x) = \max(-0.5x + 0.875, b(x))$$

$$\lambda = 6.5$$

period $= 2.0606$

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Simulations: Thacker's Oscillations

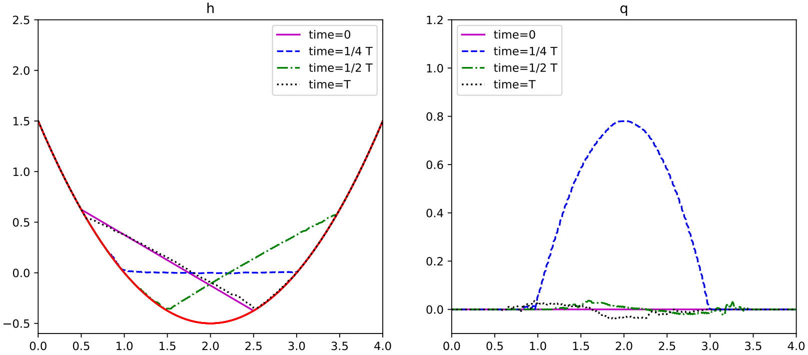


Figure: Thacker oscillations in parabola test: η^ε and $h^\varepsilon v^\varepsilon$ with $\mathbb{C}^3$ and $N = 150$

$$b(x) = (x-2)^2 - 0.5$$

$$\eta^\varepsilon(0,x) = \max(-0.5x + 0.875, b(x))$$

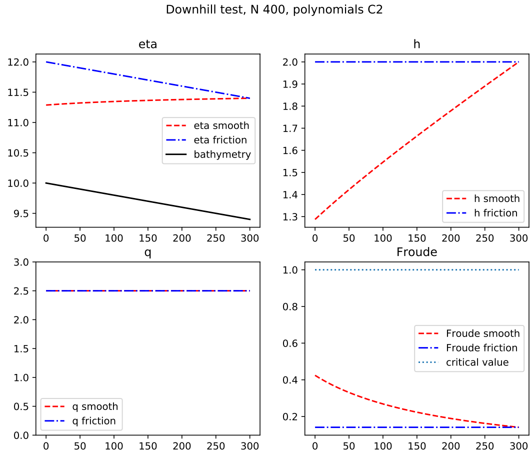
$$\lambda = 6.5$$

$$\text{period} = 2.0606$$

$$T = 5 \cdot 2.0606$$

$$\varepsilon = 10^{-14}$$

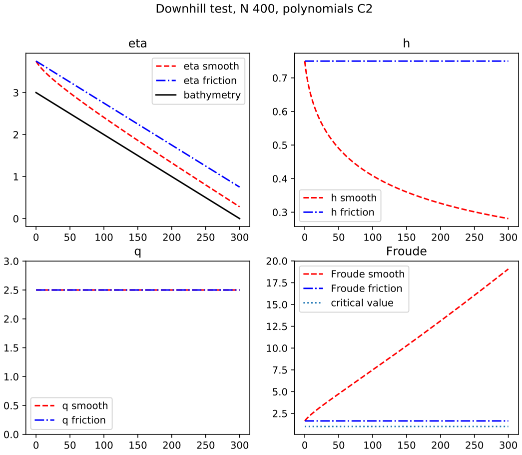
Inclined plane with friction



$$\partial_x b(x) \equiv 0.002$$
$$\eta^\varepsilon(0,x) = 2$$
$$q^\varepsilon(0,x) = 2.75$$
$$q^\varepsilon(0,t) = 2.5$$
$$h^\varepsilon(300,t) = 2$$
$$\lambda = 22$$
$$n = 0.2515597$$
$$T = 1000$$
$$\varepsilon = 10^{-14}$$

Figure: Subcritical flow: η^ε and $h^\varepsilon v^\varepsilon$ with $\mathbb{C}^2$ and $N = 400$

Inclined plane with friction



$$\partial_x b(x) = 0.01$$
$$\eta^\varepsilon(0,x) = 0.75$$
$$q^\varepsilon(0,x) = 2.75$$
$$h^\varepsilon(0,t) = 0.75$$
$$q^\varepsilon(0,t) = 2.5$$
$$\lambda = 22$$
$$n = 0.067820251$$
$$T = 1000$$
$$\varepsilon = 10^{-14}$$

Figure: Supercritical flow: η^ε and $h^\varepsilon v^\varepsilon$ with $\mathbb{C}^2$ and $N = 400$

Relaxation as diffusion

- Smooth problems
- Set $\varepsilon \approx \Delta t^{p+1}$
- Relaxation term is diffusive in Chapman–Eskong
- Pure Galerkin discretization

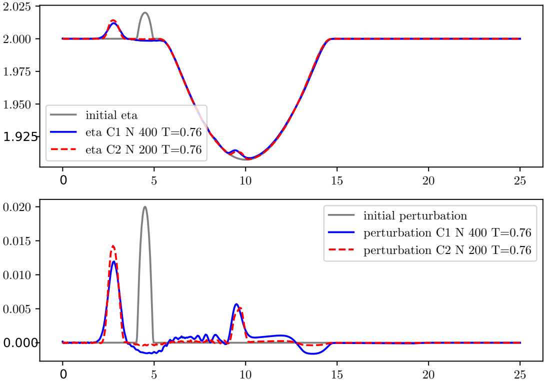


Figure: Subcritical flow $\mathbb{C}^1$ N=400, $\mathbb{C}^2$ N=200. η^ε and $h_p^\varepsilon - h^\varepsilon$ $T = 0.76$.

Relaxation as Diffusion

Relaxation as diffusion

- Smooth problems
- Set $\varepsilon \approx \Delta t^{p+1}$
- Relaxation term is diffusive in Chapman–Eskong
- Pure Galerkin discretization

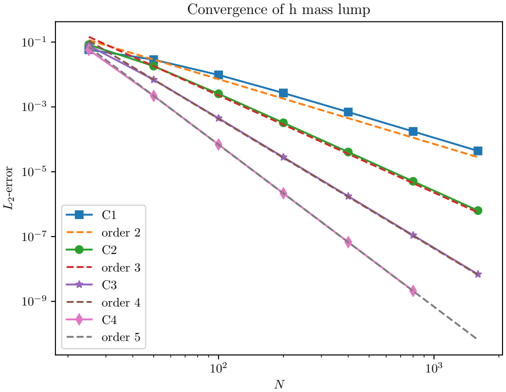


Figure: Subcritical flow $\mathbb{C}^1$ N=200, $\mathbb{C}^2$ N=100 and N=400 Perturbations of bump test cases with *cubature* elements: top to down η^ε, $h_p^\varepsilon - h^\varepsilon$ and q^ε; left $T = 0.76$ right $T = 2$.

Outline

Conclusion and perspective

Conclusions

- Kinetic Model with extra source terms (bathymetry and friction)
- Chapman–Enskog: diffusive, but extra term
- Discretization:
  - IMEX in time
  - Residual distribution
  - Deferred Correction
  - Well balanced for late at rest
  - Wet and dry area
  - Positive water height

Perspective

- MOOD
- Entropy stability
- Multi dimension

IMEX DeC RD – Bibliography

1. R. Abgrall, and D.T.. High Order Asymptotic Preserving Deferred Correction Implicit-Explicit Schemes for Kinetic Models. SIAM Journal on Scientific Computing, 42(3):B816–B845, 2020.
2. D. Aregba-Driollet, and R. Natalini. Discrete kinetic schemes for multidimensional systems of conservation laws. SIAM J. Numer. Anal., 37(6):1973–2004, 2000.
3. A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. BIT Numerical Mathematics, 40(2):241–266, 2000.
4. R. Abgrall. High Order Schemes for Hyperbolic Problems Using Globally Continuous Approximation and Avoiding Mass Matrices. Journal of Scientific Computing, 73(2):461–494, 2017.
5. M. Ricchiuto, and A. Bollermann. Stabilized residual distribution for shallow water simulations. Journal of Computational Physics, 228(4):1071–1115, 2009.

Thank you for the attention!

Residual Distribution - Choice of the scheme

(8) $$\phi_\sigma^{K,LxF}(U_h) = \int_K \varphi_\sigma \left(\nabla \cdot A(U_h) - S(U_h)\right) dx + \alpha_K (U_\sigma - \overline{U}_h^K),$$

where $\overline{U}_h^K$ is the average of U_h over the cell K and α_K is defined as

(9) $$\alpha_K = \max_{e \text{ edge } \in K} \left(\rho_S \left(\nabla A(U_h) \cdot \mathbf{n}_e \right) \right),$$

ρ_S is the spectral radius.
For monotonicity near strong discontinuities, PSI limiter:

(10) $$\beta_\sigma^K(U_h) = \max\left(\frac{\Phi_\sigma^{K,LxF}}{\Phi^K}, 0 \right) \left(\sum_{j \in K} \max\left(\frac{\Phi_j^{K,LxF}}{\Phi^K}, 0 \right) \right)^{-1}$$

Residual Distribution - Choice of the scheme

Blending between LxF and PSI:

(11)
$$\phi_\sigma^{*,K} = (1-\Theta)\beta_\sigma^K \phi_\sigma^K + \Theta \Phi_\sigma^{K,LxF},$$
$$\Theta = \frac{|\Phi^K|}{\sum_{j\in K} |\Phi_j^{K,LxF}|}.$$

Nodal residual is finally given by

(12)
$$\phi_\sigma^K = \phi_\sigma^{*,K} + \sum_{e|\text{edge of } K} \theta h_e^2 \int_e [\nabla U_h]\cdot[\nabla \varphi_\sigma] d\Gamma.$$

DeC – Proof

Proof.

Let U^* be the solution of $\mathcal{L}^2(U^*)=0$. We know that $\mathcal{L}^1(U^*)=\mathcal{L}^1(U^*)-\mathcal{L}^2(U^*)$, so that

$$\begin{aligned}
\mathcal{L}^1(U^{(k+1)})-\mathcal{L}^1(U^*) &= \left(\mathcal{L}^1(U^{(k)})-\mathcal{L}^2(U^{(k)})\right)-\left(\mathcal{L}^1(U^*)-\mathcal{L}^2(U^*)\right)\\
&=\left(\mathcal{L}^1(U^{(k)})-\mathcal{L}^1(U^*)\right)-\left(\mathcal{L}^2(U^{(k)})-\mathcal{L}^2(U^*)\right)\\
\alpha_1||U^{(k+1)}-U^*|| \leq& ||\mathcal{L}^1(U^{(k+1)})-\mathcal{L}^1(U^*)||=\\
=&||\mathcal{L}^1(U^{(k)})-\mathcal{L}^2(U^{(k)})-(\mathcal{L}^1(U^*)-\mathcal{L}^2(U^*))|| \leq\\
\leq& \alpha_2\Delta||U^{(k)}-U^*||.
\end{aligned}$$

$$||U^{(k+1)}-U^*|| \leq \left(\frac{\alpha_2}{\alpha_1}\Delta\right)||U^{(k)}-U^*|| \leq \left(\frac{\alpha_2}{\alpha_1}\Delta\right)^{k+1}||U^{(0)}-U^*||.$$

After K iteration we have an error at most of $\eta^K\cdot||U^{(0)}-U^*||$. □

Friction - Model

$$\partial_t u(x,t) + \partial_x A(u(x,t)) + \\ S_b(u(x,t)) + S_f(u(x,t)) = 0,$$

$$S_f(u) := \begin{pmatrix} 0 \\ c_f(h,q)q \end{pmatrix}.$$

Manning's law

$$c_f(h,q) = \frac{n^2 \|q\|}{h^{10/3}},$$

with n being Manning's coefficient.

$$\partial_t f^\varepsilon + \Lambda \partial_x f^\varepsilon + \tilde{S}_b(f^\varepsilon) + \tilde{S}_f(f^\varepsilon) \\ = \frac{M(Pf^\varepsilon) - f^\varepsilon}{\varepsilon},$$

with

$$\tilde{S}_f(f^\varepsilon) := \begin{pmatrix} 0 \\ c_f(h^\varepsilon, q^\varepsilon) q_1 \\ 0 \\ c_f(h^\varepsilon, q^\varepsilon) q_2 \end{pmatrix},$$

so that the projection of this source term is equal to the friction in the SW equations, i.e., $P\tilde{S}_f(f^\varepsilon) = S_f(Pf^\varepsilon)$, and it verifies also $P\Lambda\tilde{S}_f(f) = S_f(P\Lambda f)$.

Friction - Implicit Discretization

Implicit Friction, without nonlinear solver.

- limit equation: $P\mathcal{L}_1$, where $h^{\varepsilon,n+1}$ explicit and $q^{\varepsilon,n+1}$

$$|K_\sigma| \left(q_\sigma^{\varepsilon,n,m} - q_\sigma^{\varepsilon,n,0}\right) + \Delta t \beta^m \sum_{K|\sigma\in K} P\Phi_K^{\sigma,ex}(f^{n,0}) + \Delta t \beta^m |K_\sigma| S_{f,q}(u_\sigma^{\varepsilon,n,m}) = 0,$$

- $\mathcal{E}_{q,\sigma}^n =$ all the explicit terms

$$q_\sigma^{\varepsilon,n,m} \left(1 + \Delta t \beta^m \frac{n^2 |q_\sigma^{\varepsilon,n,m}|}{(h_\sigma^{\varepsilon,n,m})^{10/3}}\right) = \mathcal{E}_{q,\sigma}^n.$$

- $\Delta t, n, h_\sigma^{\varepsilon,n,m} > 0$ known, solve for the absolute value of $q_\sigma^{\varepsilon,n,m}$

$$|q_\sigma^{\varepsilon,n,m}| \left(1 + \Delta t \beta^m \frac{n^2 |q_\sigma^{\varepsilon,n,m}|}{(h_\sigma^{\varepsilon,n,m})^{10/3}}\right) = |\mathcal{E}_{q,\sigma}^n|,$$

- Only 1 positive solution $|q_\sigma^{\varepsilon,n,m}| \Longrightarrow q_\sigma^{\varepsilon,n,m}$.

Friction - Implicit Discretization

- Only 1 positive solution $|q_\sigma^{\varepsilon,n,m}| \implies q_\sigma^{\varepsilon,n,m}$.
- Kinetic model

$$|K_\sigma|(f_\sigma^{n,m} - f_\sigma^{n,0}) + \Delta t \beta^m \left(\sum_{K|\sigma\in K} \Phi_K^{\sigma,ex}(f^{n,0}) + |K_\sigma| \left(\tilde{S}_f(f_\sigma^{n,m}) + \frac{f_\sigma^{n,m} - M(Pf_\sigma^{n,m})}{\varepsilon} \right) \right) = 0,$$

- Friction coefficient $c_{f,\sigma}^{n,m} := c_f(h_\sigma^{\varepsilon,n,m}, q_\sigma^{\varepsilon,n,m})$ known, $\mathcal{E}_\sigma^n$ all the explicit terms,

$$\begin{pmatrix} h_{1,\sigma}^{n,m}\left(1+\frac{\Delta t\beta^m}{\varepsilon}\right) \\ q_{1,\sigma}^{n,m}\left(1+\frac{\Delta t\beta^m}{\varepsilon}+\Delta t\beta^m c_{f,\sigma}^{n,m}\right) \\ h_{2,\sigma}^{n,m}\left(1+\frac{\Delta t\beta^m}{\varepsilon}\right) \\ q_{2,\sigma}^{n,m}\left(1+\frac{\Delta t\beta^m}{\varepsilon}+\Delta t\beta^m c_{f,\sigma}^{n,m}\right) \end{pmatrix} - \Delta t\beta^m \frac{M(u_\sigma^{\varepsilon,n,m})}{\varepsilon} + \mathcal{E}_\sigma^n = 0.$$

Again, also this final step can be computed explicitly and, hence, all the variables $f_\sigma^{n,m}$ can be obtained efficiently.

Global flux (GF) formulation

Global flux:

- Cheng et al.,Journal of Scientific Computing, 80 (2019)
- Liu et al., SIAM Journal on Scientific Computing, 42 (2020)

Main Idea

$$
\begin{aligned}
&u_t + \partial_x A(u) = S(u)\\
&u_t + \partial_x G(u) = 0\\
&\partial_x G(u) = \partial_x A(u) - S(u)\\
&G(u(x),x) = \int_{x_0}^{x} \partial_x A(u) - S(u,s)ds
\end{aligned}
$$

Global flux (GF) formulation

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- Cheng et al.,Journal of Scientific Computing, 80 (2019)
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Main Idea

$$u_t + \partial_x A(u) = S(u)$$

$$u_t + \partial_x G(u) = 0$$

$$\partial_x G(u) = \partial_x A(u) - S(u)$$

$$G(u(x), x) = \int_{x_0}^{x} \partial_x A(u) - S(u, s) ds$$

RD as GF

$$\text{RD} \begin{cases} K = [a, b] \text{ with } p+1 \text{ degrees of freedom} \\ \phi_i^K = \int_K \omega_i (\partial_x A(u) + S(u)) \mathrm{d}x, \; 0 \leq i \leq p, \\ \sum_{i \in K} \omega_i \equiv 1 \\ \partial_t u_i = -\frac{1}{\Delta x} \sum_K \phi_i^K, \end{cases}$$

- $S_h(u) \in \mathbb{P}^{p-1}(K)$,
- $\Sigma(x) := \int_{x_0}^{x} S_h(u(s)) \mathrm{d}s \in \mathbb{P}^p(K)$,
- $\partial_x \Sigma(x) = S_h(u(x))$,
- $G(u) := A(u) - \Sigma(u)$.

Equivalence between RD and GF

Search: Global numerical flux $\hat{G}_{i+\frac{1}{2}}$ for $i = 0, \dots, p-1$

(13) $$\begin{cases} \phi_i^K = \hat{G}_{i+\frac{1}{2}} - \hat{G}_{i-\frac{1}{2}}, & i = 1, \dots, p-1, \\ \phi_0^K = \hat{G}_{\frac{1}{2}} - G(a), \\ \phi_p^K = G(b) - \hat{G}_{p-\frac{1}{2}}. \end{cases}$$

Update formula

$$\partial_t u_i = -\frac{1}{\Delta x} \sum_K \phi_i^K$$

$$\partial_t u_i = -\frac{\hat{G}_{i+\frac{1}{2}} - \hat{G}_{i-\frac{1}{2}}}{\Delta x}$$

Solution for numerical global flux

- $p+1$ equations p unknowns
- One linear dependent equation as $\phi^K = G(b) - G(a)$
- Explicit solution $\hat{G}_{\frac{1}{2}} = G(a) + \phi_0^K$; $\hat{G}_{i+\frac{1}{2}} = \hat{G}_{i-\frac{1}{2}} + \phi_i^K$.

Consistency of RD GF

$$G(a) = \hat{G}_{i+\frac{1}{2}} = G(b), \ \forall i$$
$$\Longleftrightarrow$$
$$\phi_i^K = 0, \ \forall i$$