Stability of implicit and IMEX ADER and DeC

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Workshop "High Order Structure-Preserving Semi-Implicit Schemes for Hyperbolic Equations"
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DeC and ADER properties

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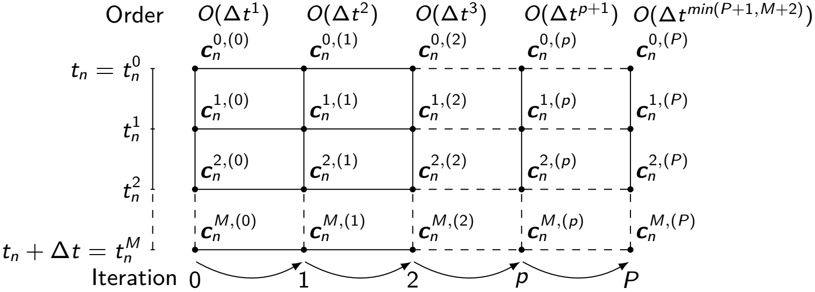
- Iterative methods
- Explicit, implicit, IMEX versions

DeC and ADER: arbitrary high order methods

DeC/ADER discretization and iterations

$$\boldsymbol{c}(t_n) \approx \boldsymbol{c}_n \qquad \boldsymbol{c}(t) = \sum_{m=0}^{M} \varphi_n^m(t)\boldsymbol{c}_n^m \quad t \in [t_n, t_{n+1}] \Longrightarrow \boldsymbol{c}_{n+1} \approx \boldsymbol{c}(t_{n+1})$$



DeC and ADER for ODEs [1,2]

$$\begin{array}{cc}
t_n^0 & \boldsymbol{c}_n^0 \\
t_n^1 & \boldsymbol{c}_n^1 \\
t_n^2 & \boldsymbol{c}_n^2 \\
t_n^{m-1} & \boldsymbol{c}_n^{m-1} \\
t_n^m & \boldsymbol{c}_n^m \\
t_n^M & \boldsymbol{c}_n^M
\end{array}$$

$$\frac{d}{dt}\boldsymbol{c}(t) = \mathbf{G}(t, \boldsymbol{c}(t)), \qquad \boldsymbol{c}_n \approx \boldsymbol{c}(t_n), \qquad \boldsymbol{c}(t) = \sum_{m=0}^{M} \varphi_n^m(t)\boldsymbol{c}_n^m \qquad \forall t \in [t_n, t_{n+1}]$$

[1] M. Han Veiga, P. Öffner, and D. T.. "DeC and ADER: Similarities, Differences and a Unified Framework." JSC, 87, 2 (2021)
[2] M. Han Veiga, L. Micalizzi and D. T.. "On improving the efficiency of ADER methods." AMC, 466, page 128426, (2024)

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DeC high order operator

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^m - \boldsymbol{c}_n - \int_{t_n^0}^{t_n^m} \mathbf{G}(\boldsymbol{c}(t))dt = 0 \quad \forall m \in [\![1, M]\!]$$

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$$\mathcal{L}^{2,m}(\underline{\mathbf{c}}) = \mathbf{c}_n^m - \mathbf{c}_n - \Delta t \sum_{r=0}^{M} \theta_r^m \mathbf{G}(\mathbf{c}_n^r) = 0 \quad \forall m \in [\![1, M]\!]$$

- Based on integral formulation
- Collocation methods
- Implicit RK with full A
- Difficult to solve directly

- Choice on points/basis functions
- Gauss–Lobatto $\Longrightarrow$ Lobatto IIIA
- High order of accuracy
  - for Lobatto $2M$

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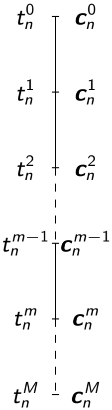

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ADER high order operator

$$\forall m \in [\![0, M]\!],$$
$$\int_{t_n}^{t_{n+1}} \varphi_n^m(t)\partial_t \boldsymbol{c}(t)dt - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\mathbf{G}(\boldsymbol{c}(t))dt = 0$$

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ADER high order operator

$$\forall m \in [\![0, M]\!], \quad \mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := A^{m,r}\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - R^{m,r}\mathbf{G}(\boldsymbol{c}_n^r) =$$

$$\varphi_n^m(t_{n+1})\varphi_n^r(t_{n+1})\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - \int_{t_n}^{t_{n+1}} \partial_t\varphi_n^m(t)\varphi_n^r(t)dt\,\boldsymbol{c}_n^r - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\varphi_n^r(t)dt\,\mathbf{G}(\boldsymbol{c}_n^r) = 0$$

- Based on weak formulation
- Integration by parts
- Implicit RK with full A
- Difficult to solve directly

- Gauss–Lobatto $\Longrightarrow$ Lobatto IIIC
- High order of accuracy
  - for Lobatto $2M$
  - for Gauss–Legendre $2M+1$

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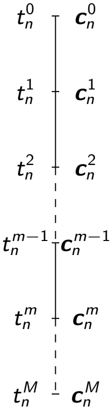
Examples of $\mathcal{L}^2$



DeC $\mathcal{L}^2$ operator

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^m - \boldsymbol{c}_n^0 - \Delta t \sum_{r=0}^{M} \theta_r^m \mathbf{G}(\boldsymbol{c}_n^r) = 0 \quad \forall m \in [\![1, M]\!]$$

$M+1=2$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \boldsymbol{c}_n^1 - \boldsymbol{c}_n^0 - \Delta t \frac{1}{2}\left(\mathbf{G}(\boldsymbol{c}_n^0) + \mathbf{G}(\boldsymbol{c}_n^1)\right)$$

$$\begin{array}{c|cc} 0 & & \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \end{array}$$

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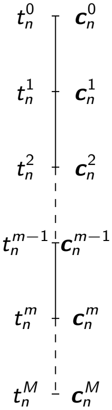

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$M+1=3$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\mathbf{c}}) = \begin{pmatrix} \mathbf{c}_n^1 - \mathbf{c}_n - \Delta t \frac{1}{2}\left(\frac{5}{24}\mathbf{G}(\mathbf{c}_n^0) + \frac{1}{3}\mathbf{G}(\mathbf{c}_n^1) - \frac{1}{24}\mathbf{G}(\mathbf{c}_n^2)\right) \\ \mathbf{c}_n^2 - \mathbf{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\mathbf{c}_n^0) + \frac{2}{3}\mathbf{G}(\mathbf{c}_n^1) + \frac{1}{6}\mathbf{G}(\mathbf{c}_n^2)\right) \end{pmatrix}$$

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Examples of $\mathcal{L}^2$

$$
\begin{array}{cc}
t_n^0 & \boldsymbol{c}_n^0 \\
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t_n^{m-1} & \boldsymbol{c}_n^{m-1} \\
t_n^m & \boldsymbol{c}_n^m \\
t_n^M & \boldsymbol{c}_n^M
\end{array}
$$

ADER $\mathcal{L}^2$ operator

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$$

$$
\varphi_n^m(t_{n+1})\varphi_n^r(t_{n+1})\boldsymbol{c}_n^r - \varphi_n^m(t_n)\boldsymbol{c}_n - \int_{t_n}^{t_{n+1}} \partial_t\varphi_n^m(t)\varphi_n^r(t)dt\, \boldsymbol{c}_n^r - \int_{t_n}^{t_{n+1}} \varphi_n^m(t)\varphi_n^r(t)dt\, \mathbf{G}(\boldsymbol{c}_n^r) = 0
$$

Examples of $\mathcal{L}^2$

t_n^0 — $\mathbf{c}_n^0$

t_n^1 — $\mathbf{c}_n^1$

t_n^2 — $\mathbf{c}_n^2$

t_n^{m-1} — $\mathbf{c}_n^{m-1}$

t_n^m — $\mathbf{c}_n^m$

t_n^M — $\mathbf{c}_n^M$

ADER $\mathcal{L}^2$ operator

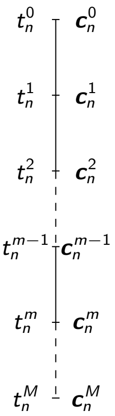
$$\forall m \in [\![0, M]\!], \quad \mathcal{L}^{2,m}(\underline{\mathbf{c}}) := \mathbf{c}_n^m - \mathbf{c}_n - (A^{-1})_{m,\ell} R^{\ell,r} \mathbf{G}(\mathbf{c}_n^r) = 0$$

$M+1=2$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\mathbf{c}}) = \begin{pmatrix} \mathbf{c}_n^0 - \mathbf{c}_n - \Delta t \left(\frac{1}{2}\mathbf{G}(\mathbf{c}_n^0) - \frac{1}{2}\mathbf{G}(\mathbf{c}_n^1)\right) \\ \mathbf{c}_n^1 - \mathbf{c}_n - \Delta t \left(\frac{1}{2}\mathbf{G}(\mathbf{c}_n^0) + \frac{1}{2}\mathbf{G}(\mathbf{c}_n^1)\right) \end{pmatrix}$$

$$\begin{array}{c|cc} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \end{array}$$

Examples of $\mathcal{L}^2$



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$M+1=3$ Gauss–Lobatto nodes

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^0 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) - \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{5}{12}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{12}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix}$$

$$\begin{array}{c|ccc} 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Properties of $\mathcal{L}^2 = 0$

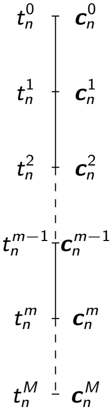
Method	DeC		ADER		
Nodes	Equispaced	Gauss–Lobatto	Equispaced	Gauss–Lobatto	Gauss–Legendre
Order	$M+1$	$2M$	$M+1$	$2M$	$2M+1$[3]
Known method	Collocation	Lobatto IIIA		Lobatto IIIC	
A–stability	☹	☺	☹	☺	☺[4]

ADER with **modal** polynomials of degree p is equivalent to ADER with Lagrange basis functions defined on points given by the underlying quadrature, if exact, Gauss–Legendre.

Implicit ADER with DG spatial discretization and Rusanov/upwind numerical flux (method of lines) is **unconditionally stable** independently on the CFL.

[3] M. Han Veiga, L. Micalizzi and D. T.. "On improving the efficiency of ADER methods." AMC, 466, page 128426, (2024)
[4] P. Öffner, L. Petri, D.T.. "Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations" (2024)

DeC and ADER operators



DeC operators

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := \boldsymbol{c}_n^m - \boldsymbol{c}_n^0 - \sum_{r=0}^{M} \int_{t_n^0}^{t_n^m} \varphi_n^r(t) dt\, \mathbf{G}(\boldsymbol{c}_n^r) = 0, \qquad \forall m \in [\![1, M]\!],$$

ADER operators

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{c}}) := A^{m,r} \boldsymbol{c}_n^r - \varphi_n^m(t_n) \boldsymbol{c}_n - \int_{t_n}^{t_{n+1}} \varphi_n^m(t) \varphi_n^r(t) dt\, \mathbf{G}(\boldsymbol{c}_n^r) = 0, \qquad \forall m \in [\![0, M]\!],$$

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$$\mathcal{L}^{1,m}(\underline{\boldsymbol{c}}) := \boldsymbol{c}_n^m - \boldsymbol{c}_n^0 - \int_{t_n^0}^{t_n^m} 1 dt\, \mathbf{G}(\boldsymbol{c}_n) = 0, \qquad \forall m \in [\![1, M]\!].$$

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Deferred Correction Iterative procedure

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

- $\mathcal{L}^1(\underline{\boldsymbol{c}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{c}}) = 0$, high order $Q(= 2M)$.

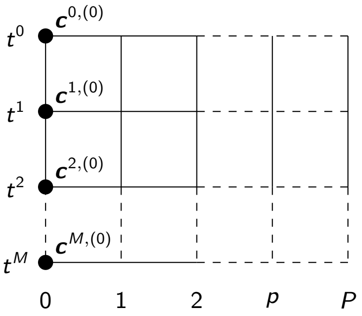
$$\boldsymbol{c}^{m,(0)} := \boldsymbol{c}(t_n), \quad m = 0, \dots, M$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

DeC Theorem

- $\mathcal{L}^1$ coercive with constant $\mathcal{O}(1)$
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

DeC converges and $\min(P, Q)$ is the order of accuracy.



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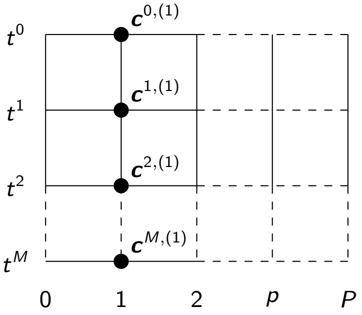

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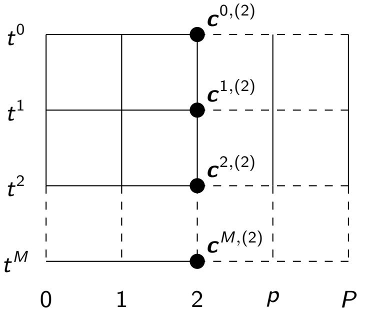

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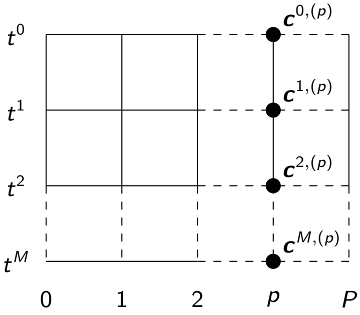

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- $\mathcal{L}^2(\underline{\boldsymbol{c}}) = 0$, high order $Q(= 2M)$.

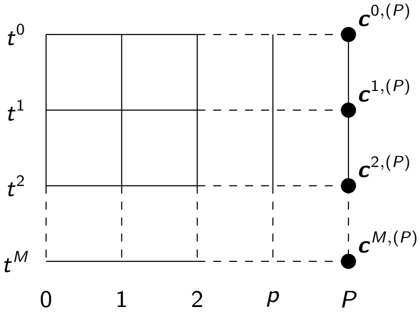
$$\boldsymbol{c}^{m,(0)} := \boldsymbol{c}(t_n), \quad m = 0, \dots, M$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

DeC Theorem

- $\mathcal{L}^1$ coercive with constant $\mathcal{O}(1)$
- $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

DeC converges and $\min(P, Q)$ is the order of accuracy.



Example of explicit DeC $M = 2$ $P = 3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24} \mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3} \mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24} \mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6} \mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3} \mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6} \mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix}$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2)\right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2)\right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

$$\star \boldsymbol{c}_n^{(0),0} = \boldsymbol{c}_n^{(0),1} = \boldsymbol{c}_n^{(0),2} = \boldsymbol{c}_n^{(1),0} = \boldsymbol{c}_n^{(2),0} = \boldsymbol{c}_n^{(3),0} = \boldsymbol{c}_n$$

Example of explicit DeC $M = 2$ $P = 3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

$$\star \boldsymbol{c}_n^{(0),0} = \boldsymbol{c}_n^{(0),1} = \boldsymbol{c}_n^{(0),2} = \boldsymbol{c}_n^{(1),0} = \boldsymbol{c}_n^{(2),0} = \boldsymbol{c}_n^{(3),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}_n^{(1),1} - \boldsymbol{c}_n^{(1),0} - \Delta t \mathbf{G}(\boldsymbol{c}_n^{(1),0}) = \boldsymbol{c}_n^{(0),1} - \boldsymbol{c}_n^{(0),0} - \Delta t \mathbf{G}(\boldsymbol{c}_n^{(0),0}) -$$

$$\boldsymbol{c}_n^{(0),1} + \boldsymbol{c}_n^{(0),0} + \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^{(0),0}) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^{(0),1}) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^{(0),2}) \right)$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

$$\star \boldsymbol{c}_n^{(0),0} = \boldsymbol{c}_n^{(0),1} = \boldsymbol{c}_n^{(0),2} = \boldsymbol{c}_n^{(1),0} = \boldsymbol{c}_n^{(2),0} = \boldsymbol{c}_n^{(3),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}_n^{(1),1} = \boldsymbol{c}_n + \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n)$$

$$\begin{array}{c|c} 0 & \\ \frac{1}{2} & \frac{1}{2} \\ & \\ & \\ \hline & \end{array}$$

Example of explicit DeC $M = 2$ $P = 3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

$$\star \boldsymbol{c}_n^{(0),0} = \boldsymbol{c}_n^{(0),1} = \boldsymbol{c}_n^{(0),2} = \boldsymbol{c}_n^{(1),0} = \boldsymbol{c}_n^{(2),0} = \boldsymbol{c}_n^{(3),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}_n^{(1),1} = \boldsymbol{c}_n + \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}_n^{(1),2} = \boldsymbol{c}_n + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$\begin{array}{c|c} 0 & \\ \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \\ & \\ & \\ \hline & \end{array}$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

$$\star \boldsymbol{c}_n^{(0),0} = \boldsymbol{c}_n^{(0),1} = \boldsymbol{c}_n^{(0),2} = \boldsymbol{c}_n^{(1),0} = \boldsymbol{c}_n^{(2),0} = \boldsymbol{c}_n^{(3),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}_n^{(1),1} = \boldsymbol{c}_n + \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}_n^{(1),2} = \boldsymbol{c}_n + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}_n^{(2),1} = \boldsymbol{c}_n + \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^{(1),0}) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^{(1),1}) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^{(1),2}) \right)$$

$$\star \boldsymbol{c}_n^{(2),2} = \boldsymbol{c}_n + \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(1),0}) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^{(1),1}) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(1),2}) \right)$$

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & & & \end{array}$$

Example of explicit DeC $M=2$ $P=3$

$$\mathcal{L}^2(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^1) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^2) \right) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^0) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^1) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^2) \right) \end{pmatrix} \qquad \mathcal{L}^1(\underline{\boldsymbol{c}}) = \begin{pmatrix} \boldsymbol{c}_n^1 - \boldsymbol{c}_n - \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n^0) \\ \boldsymbol{c}_n^2 - \boldsymbol{c}_n - \Delta t \mathbf{G}(\boldsymbol{c}_n^0) \end{pmatrix}$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)}), \qquad p = 1, \dots, 3.$$

$$\star \boldsymbol{c}_n^{(0),0} = \boldsymbol{c}_n^{(0),1} = \boldsymbol{c}_n^{(0),2} = \boldsymbol{c}_n^{(1),0} = \boldsymbol{c}_n^{(2),0} = \boldsymbol{c}_n^{(3),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}_n^{(1),1} = \boldsymbol{c}_n + \Delta t \frac{1}{2}\mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}_n^{(1),2} = \boldsymbol{c}_n + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}_n^{(2),1} = \boldsymbol{c}_n + \Delta t \left(\frac{5}{24}\mathbf{G}(\boldsymbol{c}_n^{(1),0}) + \frac{1}{3}\mathbf{G}(\boldsymbol{c}_n^{(1),1}) - \frac{1}{24}\mathbf{G}(\boldsymbol{c}_n^{(1),2}) \right)$$

$$\star \boldsymbol{c}_n^{(2),2} = \boldsymbol{c}_n + \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(1),0}) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^{(1),1}) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(1),2}) \right)$$

$$\star \boldsymbol{c}_{n+1} = \boldsymbol{c}_n^{(3),2} = \boldsymbol{c}_n + \Delta t \left(\frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(2),0}) + \frac{2}{3}\mathbf{G}(\boldsymbol{c}_n^{(2),1}) + \frac{1}{6}\mathbf{G}(\boldsymbol{c}_n^{(2),2}) \right)$$

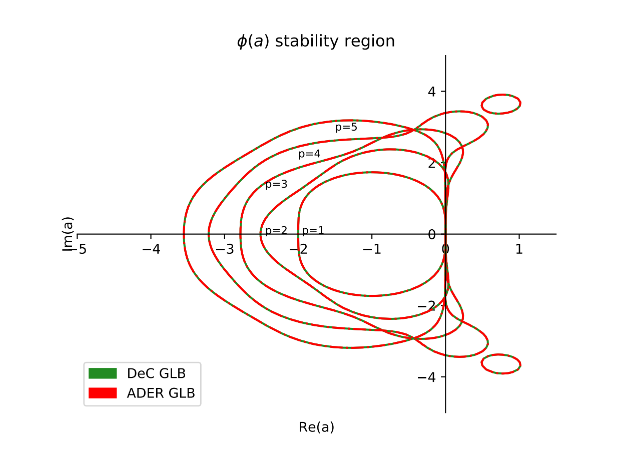
$$\begin{array}{c|ccccc} 0 & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ 1 & 1 & & & & \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} & & \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ \hline & \frac{1}{6} & & & \frac{2}{3} & \frac{1}{6} \end{array}$$

Stability of explicit DeC/ADER

Stability function

All the described DeC/ADER explicit methods of order P have stability function given by

$$R(z) = \sum_{r=0}^{P} \frac{1}{r!} z^r.$$



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Implicit DeC/ADER

Implicit Recipe

- $\mathcal{L}^1$ implicit
- Fully implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \mathbf{G}(\underline{\boldsymbol{c}})$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\boldsymbol{c}}) \qquad (= \mathcal{L}^2(\underline{\boldsymbol{c}}))$$

Implicit DeC/ADER

Implicit Recipe

- $\mathcal{L}^1$ implicit
- Fully implicit

$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \mathbf{G}(\underline{\mathbf{c}})$$
$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}}) \qquad (= \mathcal{L}^2(\underline{\mathbf{c}}))$$

- Linearly implicit (similar to Newton/Rosenbrock)

$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \left(\mathbf{G}(\mathbf{c}_n) + \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t A^{-1} R \left(\mathbf{G}(\mathbf{c}_n) + \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\mathbf{c}}^{(p)}) = \mathcal{L}^1(\underline{\mathbf{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\mathbf{c}}^{(p-1)})$$

Implicit DeC/ADER

$$\mathcal{L}^1(\underline{\mathbf{c}}^{(p)}) = \mathcal{L}^1(\underline{\mathbf{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\mathbf{c}}^{(p-1)})$$

Implicit Recipe

- $\mathcal{L}^1$ implicit
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$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \mathbf{G}(\underline{\mathbf{c}})$$
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DeC Full Implicit IMDeC

$$\underline{\mathbf{c}}^{(p)} - \underline{\mathbf{c}}^{(p-1)} + \Delta t \beta \left(\mathbf{G}(\underline{\mathbf{c}}^{(p)}) - \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})\right)$$
$$= \mathbf{c}_n - \underline{\mathbf{c}}^{(p-1)} + \Delta t \Theta \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})$$

DeC Linearly Implicit IMDeC-Lin

$$\left[I - \Delta t \beta \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)\right] \left(\underline{\mathbf{c}}^{(p)} - \underline{\mathbf{c}}^{(p-1)}\right)$$
$$= \mathbf{c}_n - \underline{\mathbf{c}}^{(p-1)} + \Delta t \Theta \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})$$

Implicit DeC/ADER

This leads to the following RK Butcher tableau

$$
\begin{array}{c|cccccccc}
0 & 0 & & & & & & & \\
\underline{\beta} & \underline{0} & \underline{\underline{B}} & & & & & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & & & & & \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & & & & \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} & \\
1 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \beta^M \\
\hline
 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \beta^M
\end{array}
$$

with $B_{mr} = \delta_{mr}\beta^m$ for $m, r = 1, \dots, M$ and δ_{mr} the Kronecker delta.

Implicit DeC/ADER

$$\mathcal{L}^1(\underline{\mathbf{c}}^{(p)}) = \mathcal{L}^1(\underline{\mathbf{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\mathbf{c}}^{(p-1)})$$

Implicit Recipe

- $\mathcal{L}^1$ implicit
- Fully implicit

$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \mathbf{G}(\underline{\mathbf{c}})$$
$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}}) \qquad (= \mathcal{L}^2(\underline{\mathbf{c}}))$$

- Linearly implicit (similar to Newton/Rosenbrock)

$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t \beta \left(\mathbf{G}(\mathbf{c}_n) + \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\mathbf{c}}) := \underline{\mathbf{c}} - \mathbf{c}_n - \Delta t A^{-1} R \left(\mathbf{G}(\mathbf{c}_n) + \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)(\underline{\mathbf{c}} - \mathbf{c}_n)\right)$$

ADER Full Implicit IMADER

$$\mathcal{L}^1 = \mathcal{L}^2$$
$$\underline{\mathbf{c}}^{(p)} - \mathbf{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}}^{(p)}) = 0$$

ADER Linearly Implicit IMADER-Lin

$$\left[I - \Delta t A^{-1} R \partial_{\mathbf{c}} \mathbf{G}(\mathbf{c}_n)\right] (\underline{\mathbf{c}}^{(p)} - \underline{\mathbf{c}}^{(p-1)})$$
$$= \mathbf{c}_n - \underline{\mathbf{c}}^{(p-1)} + \Delta t A^{-1} R \mathbf{G}(\underline{\mathbf{c}}^{(p-1)})$$

Implicit DeC/ADER

$$
\begin{array}{c|cccccc}
\underline{P} & \underline{\underline{Q}} & & & & & \\
\underline{P} & \underline{\underline{0}} & \underline{\underline{Q}} & & & & \\
\vdots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & & \\
\vdots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\
\underline{P} & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\hline
 & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{b}^T
\end{array}
$$

$^2(\underline{\boldsymbol{c}}))$

k)

$\boldsymbol{c}_n))$

$(\underline{\boldsymbol{c}} - \boldsymbol{c}_n))$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

ADER Full Implicit IMADER

$$\mathcal{L}^1 = \mathcal{L}^2$$

$$\underline{\boldsymbol{c}}^{(p)} - \boldsymbol{c}_n - \Delta t A^{-1} R \mathbf{G}(\underline{\boldsymbol{c}}^{(p)}) = 0$$

ADER Linearly Implicit IMADER-Lin

$$\left[I - \Delta t A^{-1} R \partial_{\boldsymbol{c}} \mathbf{G}(\boldsymbol{c}_n) \right] (\underline{\boldsymbol{c}}^{(p)} - \underline{\boldsymbol{c}}^{(p-1)})$$

$$= \boldsymbol{c}_n - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t A^{-1} R \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})$$

Example of IMDeC and IMDeC-Lin

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c})$$

IMDeC2

$$\star \boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}^{(1),0} = \boldsymbol{c}^{(2),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \Delta t \mathbf{G}(\boldsymbol{c}^{(1),1})$$

$$\star \boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \Delta t \left(\mathbf{G}(\boldsymbol{c}^{(2),1}) - \mathbf{G}(\boldsymbol{c}^{(1),1}) + \frac{\mathbf{G}(\boldsymbol{c}^{(1),1}) + \mathbf{G}(\boldsymbol{c}^{(1),0})}{2} \right)$$

IMDeC2-Lin

$$\star \boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}^{(1),0} = \boldsymbol{c}^{(2),0} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \Delta t \partial_{\boldsymbol{c}} \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(1),1} - \boldsymbol{c}_n) + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \Delta t \left(\partial_{\boldsymbol{c}} \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(2),1} - \boldsymbol{c}^{(1),1}) + \frac{\mathbf{G}(\boldsymbol{c}^{(1),1}) + \mathbf{G}(\boldsymbol{c}^{(1),0})}{2} \right)$$

Example of IMADER and IMADER-Lin

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c})$$

IMADER2

$$\star \boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}^{(1),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}(-\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1}))$$

$$\star \boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}(\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1}))$$

$$\star \boldsymbol{c}^{(2),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}(-\mathbf{G}(\boldsymbol{c}^{(2),0}) + \mathbf{G}(\boldsymbol{c}^{(2),1}))$$

$$\star \boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}(\mathbf{G}(\boldsymbol{c}^{(2),0}) + \mathbf{G}(\boldsymbol{c}^{(2),1}))$$

Waste of resources!

IMADER2-Lin

$$\star \boldsymbol{c}^{(0),0} = \boldsymbol{c}^{(0),1} = \boldsymbol{c}_n$$

$$\star \boldsymbol{c}^{(1),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(-\boldsymbol{c}^{(1),0} + \boldsymbol{c}^{(1),1})$$

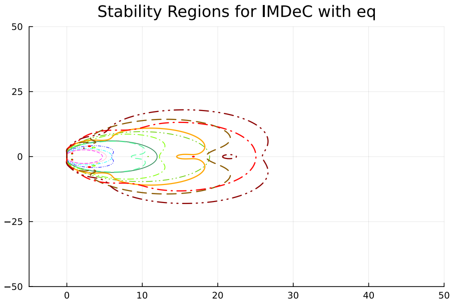
$$\star \boldsymbol{c}^{(1),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(1),0} + \boldsymbol{c}^{(1),1} - 2\boldsymbol{c}_n) + \Delta t \mathbf{G}(\boldsymbol{c}_n)$$

$$\star \boldsymbol{c}^{(2),0} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(-\boldsymbol{c}^{(2),0} + \boldsymbol{c}^{(2),1} + \boldsymbol{c}^{(1),0} - \boldsymbol{c}^{(1),1}) + \frac{\Delta t}{2}\left(-\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1})\right)$$

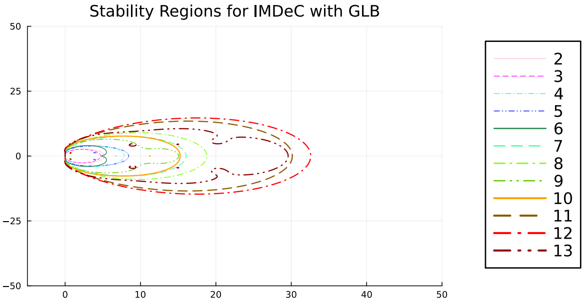
$$\star \boldsymbol{c}^{(2),1} = \boldsymbol{c}_n + \frac{\Delta t}{2}\partial_c \mathbf{G}(\boldsymbol{c}_n)(\boldsymbol{c}^{(2),0} + \boldsymbol{c}^{(2),1} - \boldsymbol{c}^{(1),0} - \boldsymbol{c}^{(1),1}) + \frac{\Delta t}{2}\left(\mathbf{G}(\boldsymbol{c}^{(1),0}) + \mathbf{G}(\boldsymbol{c}^{(1),1})\right)$$

Stability of IMDeC

A-Stable?



GLB nodes



eq nodes

Figure: ImDeC stability region for orders 2 to 13.

Almost A-Stable!

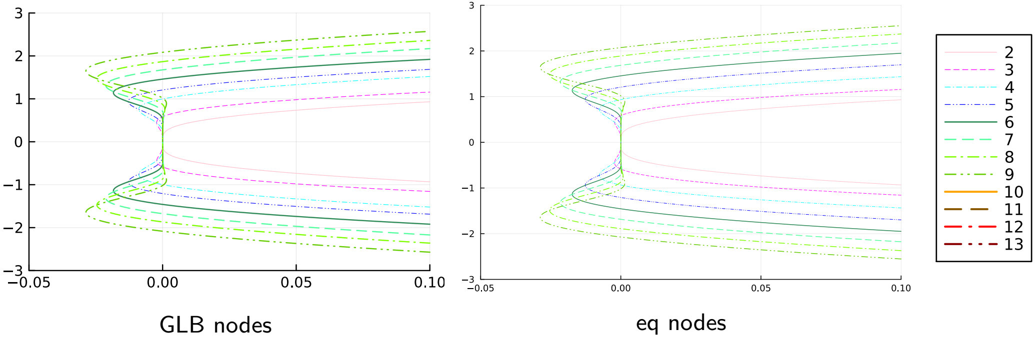


Figure: Zoomed ImDeC stability region for orders 2 to 7.

Stability of IMADER

A-Stable? GLB, GLG Yes! Proof [5], Equi Not clear

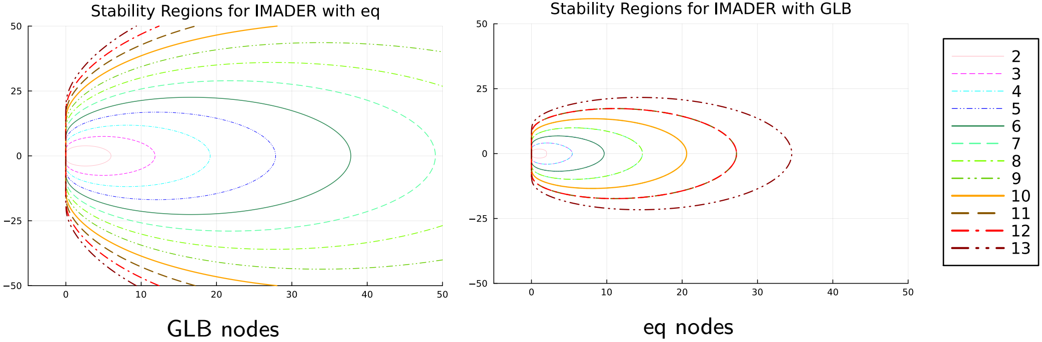


Figure: ImADER stability region for orders 2 to 13.

[5] P. Öffner, L. Petri, D.T.. "Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations" (2024)

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IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S} \cdot \boldsymbol{c}$$

IMEX Recipe

- $\mathcal{L}^1$ implicit for **S**

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S} \cdot \boldsymbol{c}$$

IMEX Recipe

- $\mathcal{L}^1$ implicit for $\mathbf{S}$
- Nonlinear implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

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$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}\cdot \boldsymbol{c}$$

IMEX Recipe

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- Nonlinear implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

- Linearly IMEX (EIN methods / Add-and-subtract)

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S} \cdot \boldsymbol{c}$$

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$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

IMEX DeC (nonlinear)

$$\underline{\boldsymbol{c}}^{(p)} - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}^{(p)}) - \mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)})\right)$$
$$= \boldsymbol{c}_n - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t \Theta (\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)}))$$
$$\Longleftrightarrow$$
$$\underline{\boldsymbol{c}}^{(p)} = \boldsymbol{c}_n + \Delta t \Big[\beta \mathbf{S}(\underline{\boldsymbol{c}}^{(p)})$$
$$+ (\Theta - \beta)\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \Theta \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\Big]$$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}\cdot \boldsymbol{c}$$

$$\begin{array}{c|ccccccccc}
0 & 0 \\
\underline{\beta} & \underline{\underline{0}} & \underline{\underline{B}} \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} - \underline{\underline{B}} & \underline{\underline{B}} \\
1 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \beta^M \\
\hline
 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\tilde{\theta}}^M - \underline{B}^M & \beta^M
\end{array},
\qquad
\begin{array}{c|cccccccc}
0 & 0 \\
\underline{\beta} & \underline{\beta} \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} \\
1 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\theta}_r^M & 0 \\
\hline
 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\theta}_r^M & 0
\end{array}.$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n) \underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n) \right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_{\boldsymbol{c}} \mathbf{S}(\boldsymbol{c}_n) \underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n) \right)$$

$$\underline{\boldsymbol{c}}^{(p)} = \boldsymbol{c}_n + \Delta t \Big[\beta \mathbf{S}(\underline{\boldsymbol{c}}^{(p)})$$
$$+ (\Theta - \beta) \mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \Theta \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)}) \Big]$$

IMEX DeC and ADER

$$\partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}(\boldsymbol{c}) \text{ or better } \partial_t \boldsymbol{c} = \mathbf{G}(\boldsymbol{c}) + \mathbf{S}\cdot \boldsymbol{c}$$

IMEX Recipe

- $\mathcal{L}^1$ implicit for $\mathbf{S}$
- Nonlinear implicit

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}) + \mathbf{G}(\boldsymbol{c}_n)\right)$$

- Linearly IMEX (EIN methods / Add-and-subtract)

$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t \beta \left(\partial_c \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$
$$\mathcal{L}^1(\underline{\boldsymbol{c}}) := \underline{\boldsymbol{c}} - \boldsymbol{c}_n - \Delta t A^{-1} R \left(\partial_c \mathbf{S}(\boldsymbol{c}_n)\underline{\boldsymbol{c}} + \mathbf{G}(\boldsymbol{c}_n)\right)$$

$$\mathcal{L}^1(\underline{\boldsymbol{c}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{c}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{c}}^{(p-1)})$$

IMEX ADER (nonlinear)

$$\underline{\boldsymbol{c}}^{(p)} - \underline{\boldsymbol{c}}^{(p-1)} - \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}^{(p)}) - \mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)})\right)$$
$$= \boldsymbol{c}_n - \underline{\boldsymbol{c}}^{(p-1)} + \Delta t A^{-1} R \left(\mathbf{S}(\underline{\boldsymbol{c}}^{(p-1)}) + \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\right)$$
$$\Longleftrightarrow$$
$$\underline{\boldsymbol{c}}^{(p)} = \boldsymbol{c}_n + \Delta t A^{-1} R \left[\mathbf{S}(\underline{\boldsymbol{c}}^{(p)}) + \mathbf{G}(\underline{\boldsymbol{c}}^{(p-1)})\right]$$

IMEX DeC and ADER

$\partial_t \underline{c} = \mathbf{G}(\underline{c}) + \mathbf{S}(\underline{c})$ or better $\partial_t \underline{c} = \mathbf{G}(\underline{c}) + \mathbf{S}\,\underline{c}$

$$
\begin{array}{c|cccccccc}
0 & 0 & & & & & & & \\
\underline{P} & \underline{0} & \underline{\underline{Q}} & & & & & & \\
\underline{P} & \underline{0} & \underline{\underline{0}} & \underline{\underline{Q}} & & & & & \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & & & \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & & \\
\underline{P} & \underline{0} & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} \\
\hline
 & \underline{0}^T & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{b}^T
\end{array},
\qquad
\begin{array}{c|ccccccc}
0 & 0 & & & & & & \\
\underline{P} & \underline{P} & & & & & & \\
\underline{P} & \underline{0} & \underline{\underline{Q}} & & & & & \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{Q}} & & & & \\
\vdots & \underline{0} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{Q}} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & \\
\underline{P} & \underline{0} & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{Q}} & \\
\hline
 & \underline{0}^T & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{b}^T
\end{array}.
$$

$$\mathcal{L}^1(\underline{c}) := \underline{c} - \mathbf{c}_n - \Delta t A^{-1} R\left(\partial_{\mathbf{c}}\mathbf{S}(\mathbf{c}_n)\underline{c} + \mathbf{G}(\mathbf{c}_n)\right)$$

Stability for IMEX methods

How to compute the stability region for IMEX methods? $\partial_t \mathbf{c} = G\mathbf{c} + S\mathbf{c}, \quad G, S \in \mathbb{C}$

$\mathbf{c}_{n+1} = R(\Delta t G, \Delta t S)\mathbf{c}_n = R(\lambda_G, \lambda_S)\mathbf{c}_n \qquad R(\cdot,\cdot) : \mathbb{C}^2 \to \mathbb{C} \qquad$ Hard to study $\{|R| \leq 1\} \subset \mathbb{C}^2$

Minion[a]

- $\lambda_G \in i\mathbb{R}$
- $\lambda_S \in \mathbb{R}$
- $R(\lambda_G, \lambda_S) : \mathbb{C} \to \mathbb{C}$
- Not really representative of high order operators
- Simple for comparisons

[a]M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. Commun. Math. Sci., 1(3):471–500, 09 2003.

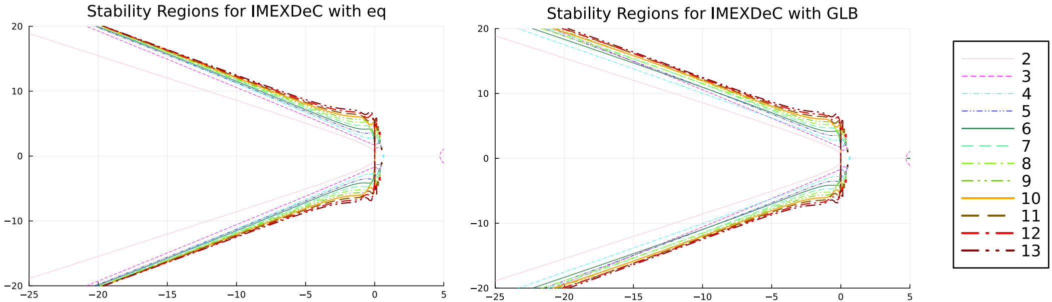
Hundsdorfer[a]

- $\mathcal{D}_0 := \{\lambda_G \in \mathbb{C} : |R(\lambda_G, \lambda_S)| \leq 1, \forall \lambda_S \in \mathbb{C}^-\}$
- $\mathcal{D}_1 := \{\lambda_S \in \mathbb{C} : |R(\lambda_G, \lambda_S)| \leq 1, \forall \lambda_G \in \mathcal{S}_0\}$
  - $\mathcal{S}_0 = \{z \in \mathbb{C} : |1 + z| \leq 1\}$
- Quite restrictive
  - $\mathcal{D}_0 = \emptyset$ often, we are asking essentially more than A–stability
- Numerical discretization more involved than Minion's one

[a]W. Hundsdorfer and J. Verwer. Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations. Springer Berlin Heidelberg, 2003.

Minion's Approach

IMEX DeC Stability Region with Minion's approach

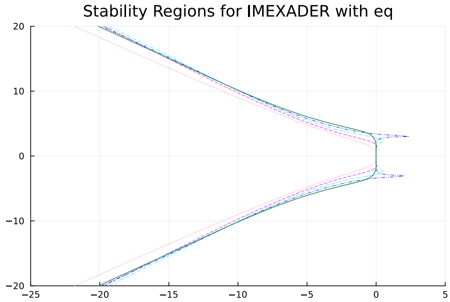


Equispaced nodes

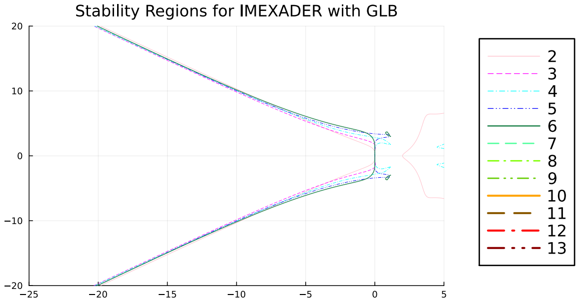
Gauss–Lobatto nodes

Minion's Approach

IMEX ADER Stability Region with Minion's approach



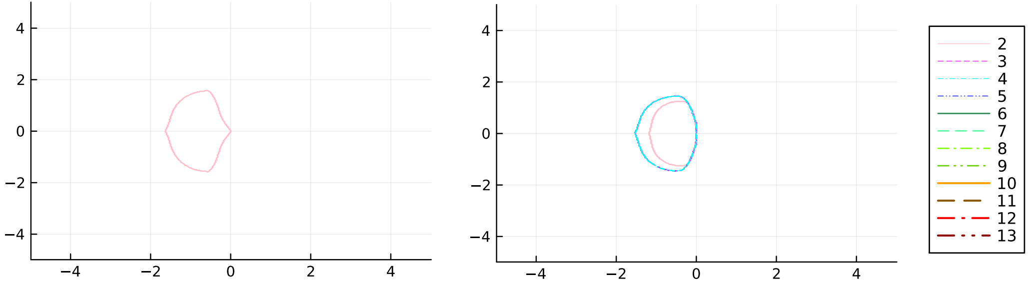
Equispaced nodes



Gauss–Lobatto nodes

Hundsdorfer's Approach

IMEX ADER Stability Region with $\mathcal{D}_0$ Hundsdorfer's approach

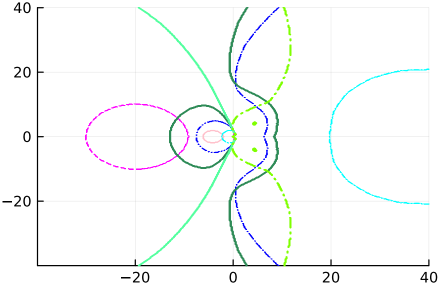


Equispaced nodes for order 2

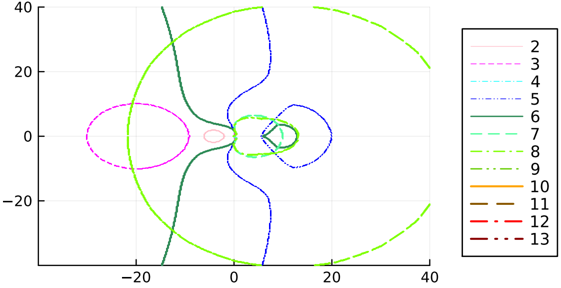
Gauss–Lobatto nodes for orders 2 to 4

Hundsdorfer's Approach

IMEX DeC Stability Region with $\mathcal{D}_1$ Hundsdorfer's approach: Bounded areas



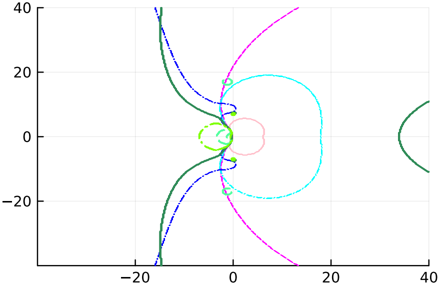
Equispaced nodes



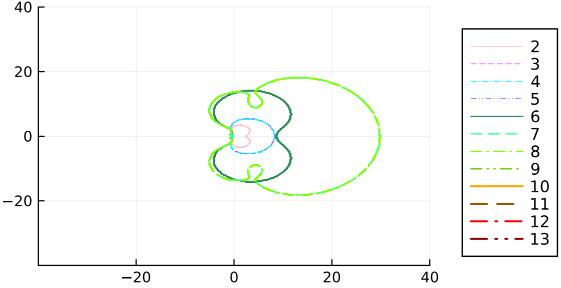
Gauss–Lobatto nodes

Hundsdorfer's Approach

IMEX ADER Stability Region with $\mathcal{D}_1$ Hundsdorfer's approach: Unbounded areas



Equispaced nodes



Gauss–Lobatto nodes

IMEX Stability Summary

Method	Minion	$\mathcal{D}_0$ Hundsdorfer	$\mathcal{D}_1$ Hundsdorfer
IMEX DeC equi	A(α)-stability $\alpha \approx 35^\circ$ Order 2 strictest stab	Always unstable	Bounded areas increasing with order
IMEX DeC GLB	$\uparrow$	Always unstable	Bounded areas increasing with order
IMEX ADER equi	$\uparrow$	Order 2 stable	Unlimited areas almost A-stable bounded for orders 5 and 8
IMEX ADER GLB	$\uparrow$	Order 2-4 stable	Unlimited areas almost A-stable

Applications

Stability of Advection-diffusion/advection-dispersion

- Identify relevant parameters, e.g. $C = c\frac{\Delta t}{\Delta x}$, $D = d\frac{\Delta t}{\Delta x^2}$, $E = \frac{C^2}{D}$.
- Draw stability region in plane $C - E$
- Find stability bounds: if $C < C_0$ or if $E < E_0$.

Approximated border values C_0 (up to 2 decimals) and E_0 (up to 1 decimal) for Gauss–Lobatto methods

Order	DeC		ADER	
	C_0	E_0	C_0	E_0
2	0.50	2.5	0.50	0.7
3	1.63	6.1	1.63	4.5
4	1.04	6.9	1.04	4.2
5	1.74	8.8	1.74	7.2
6	1.60	4.1	1.60	4.1
7	1.94	9.5	1.94	8.5
8	2.00	10.2	2.00	9.8

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- **Nonlinear stiff equations**
  - coefficients for stability (add/subtract)

Summary and Future Research

Summary

- **DeC and ADER**
- **Explicit, Implicit, IMEX, nonlinear solvers**
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- **Diffusion – Advection Equation**

Future Research

- **Nonlinear stiff equations**
  - coefficients for stability (add/subtract)
- **Implicit Advection**

THANK YOU!

Literature

- Öffner, P., Petri, L., and Torlo, D. (2025). Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations. Applied Numerical Mathematics, 212, p. 110-134.
- M. Han Veiga, L. Micalizzi and D. Torlo. (2024). On improving the efficiency of ADER methods. Applied Mathematics and Computation, 466, page 128426.
- M. H. Veiga, P. Öffner, and D. Torlo. (2021). DeC and ADER: Similarities, Differences and a Unified Framework. Journal of Scientific Computing, 87, 2.

Advection – diffusion problems

$$\partial_t u + a\partial_x u - d\partial_{xx} u = 0 \qquad a, d \geq 0$$

Discretization

- Explicit advection term $\frac{a\Delta t}{\Delta x} Du \approx \Delta t a \partial_x u$
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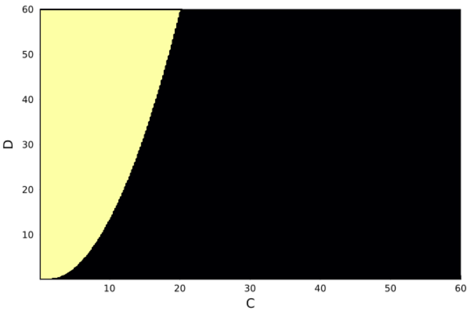

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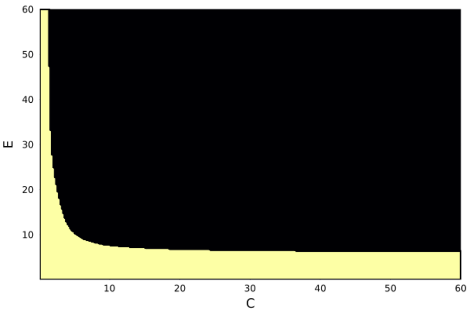


$C - E$ Stability Areas for advection–diffusion

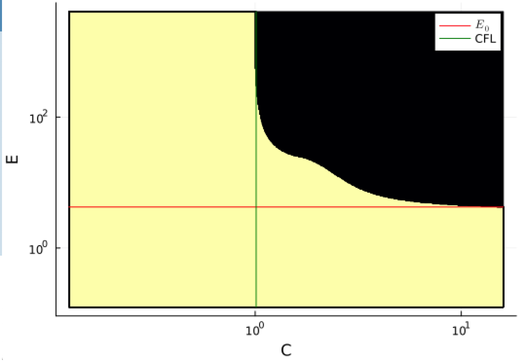
Stability region description (often)

- If $C = \frac{a\Delta t}{\Delta x} \leq C_0 \implies$ Stable
- If $E \leq E_0 \implies$ Stable

$$E = \frac{a^2 \Delta t}{d} \leq E_0 \iff \Delta t \leq \frac{E_0 d}{a^2} =: \tau_0 {}^{a}$$

  - Independent on Δx

[a] M. Tan, J. Cheng, and C.-W. Shu. Stability of high order finite difference schemes with implicit-explicit time-marching for convection-diffusion and convection-dispersion equations. International Journal of Numerical Analysis and Modeling, 18(3):362-383, 2021.



$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

- Advection $Du_j = \frac{u_j - u_{j-1}}{\Delta x}$ first order
- Diffusion $D_2 u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2}$ second order
- **Time orders** from 2 to 8

Gauss–Lobatto

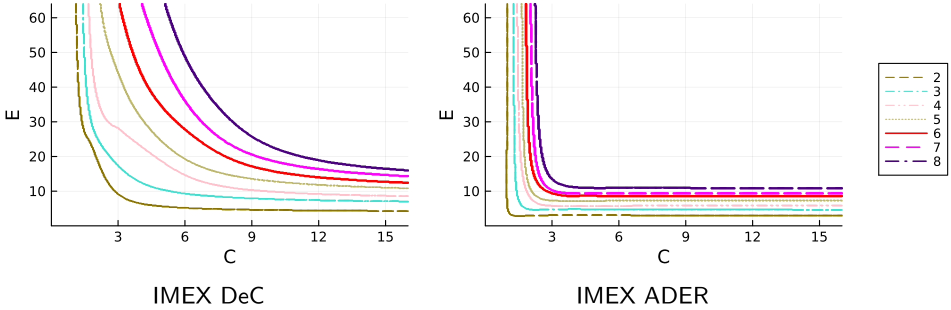


Figure: Stability areas for orders 2 to 8 with Gauss-Lobatto nodes.

$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

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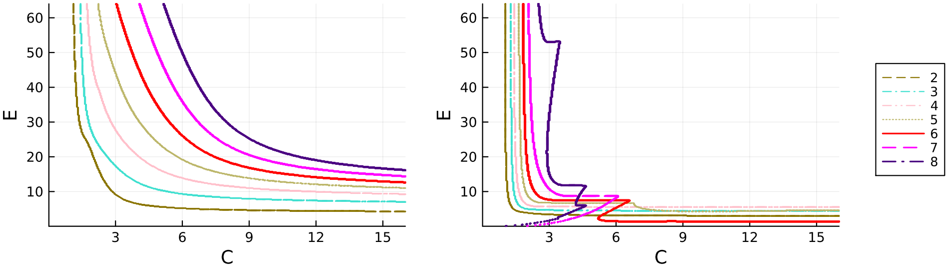


IMEX DeC

IMEX ADER

Figure: Stability areas for orders 2 to 8 with equispaced nodes.

$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

- **Advection operators** order from 1 to 8
- Diffusion $D_2 u_j = \frac{u_{j-1}-2u_j+u_{j+1}}{\Delta x^2}$ second order
- Time order 8

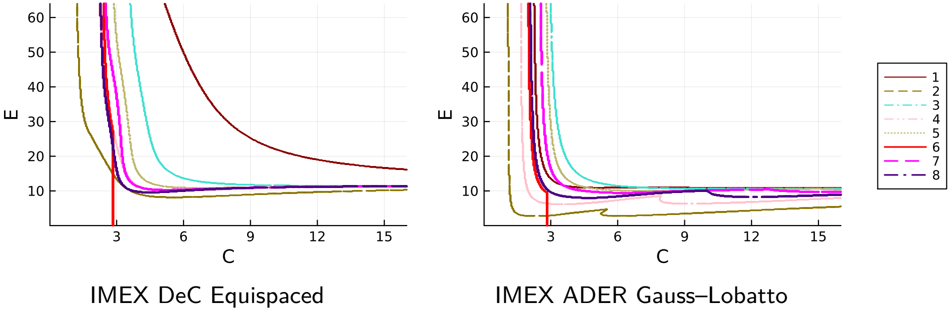


Figure: Stability areas for orders 1 to 8 of the advection operator

$C-E$ stability plots for IMEX DeC/ADER on advection–diffusion

- Advection $Du_j = \frac{u_j - u_{j-1}}{\Delta x}$ first order
- **Diffusion operators** central order in $[2,4,6,8]$
- Time order 8

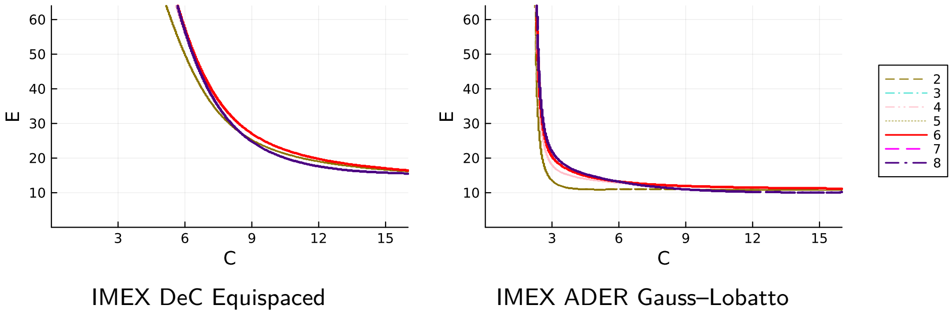
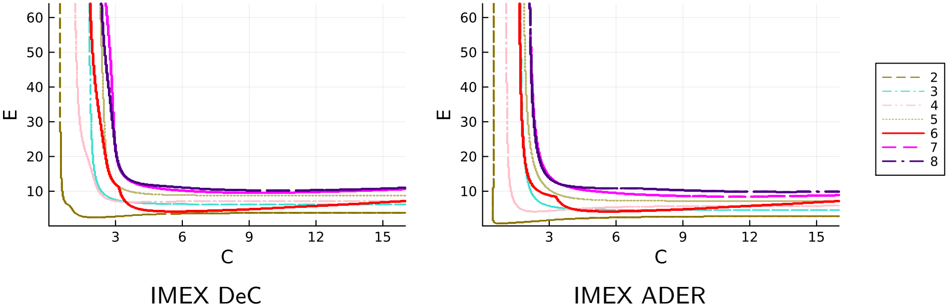


Figure: Stability areas for orders 2 to 8 of the diffusion operator

$C - E$ stability plots for IMEX DeC/ADER on advection–diffusion

- **Advection operator** order k
- **Diffusion operator** order k
- **Time order** k from 2 to 8

Gauss–Lobatto



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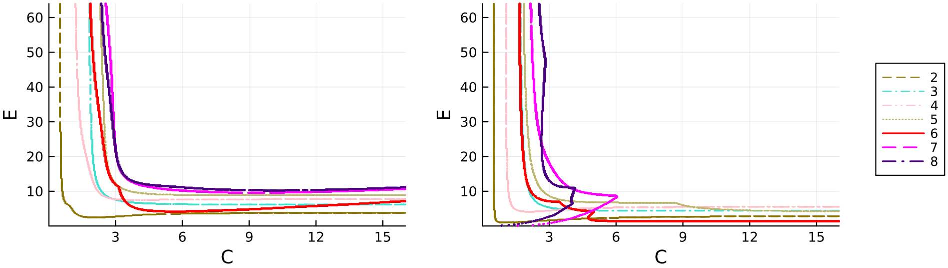
Equispaced



IMEX DeC IMEX ADER

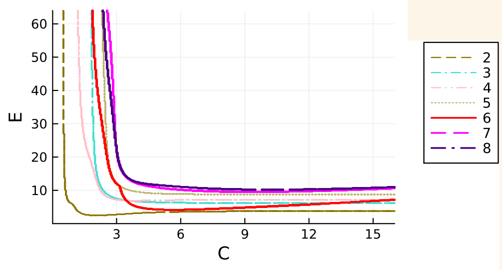
Figure: Stability areas for orders 2 to 8 with equispaced nodes.

C-E stability optimal values

Approximated border values C_0 (up to 2 decimals) and E_0 (up to 1 decimal) for Gauss–Lobatto methods

Order	DeC		ADER	
	C_0	E_0	C_0	E_0
2	0.50	2.5	0.50	0.7
3	1.63	6.1	1.63	4.5
4	1.04	6.9	1.04	4.2
5	1.74	8.8	1.74	7.2
6	1.60	4.1	1.60	4.1
7	1.94	9.5	1.94	8.5
8	2.00	10.2	2.00	9.8

IMEX DeC Gauss–Lobatto



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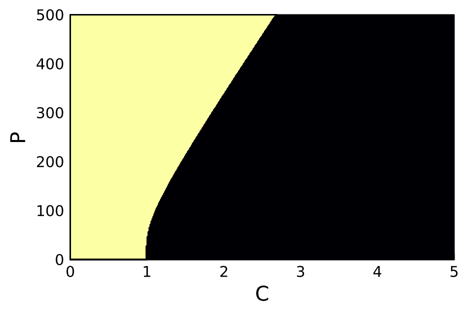

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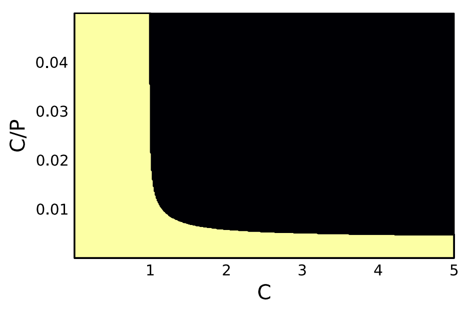

$C - E$ Stability Areas for advection–dispersion

IMEX DeC GLB 2
Advection order 1
Dispersion order 3

Stability region description

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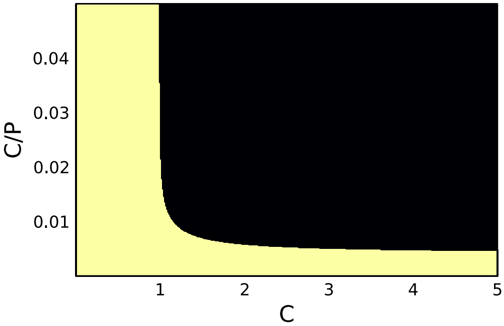

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Advection order 1
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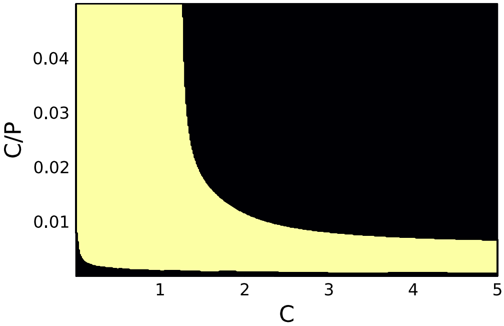

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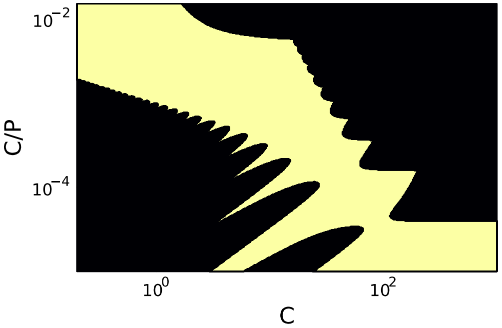
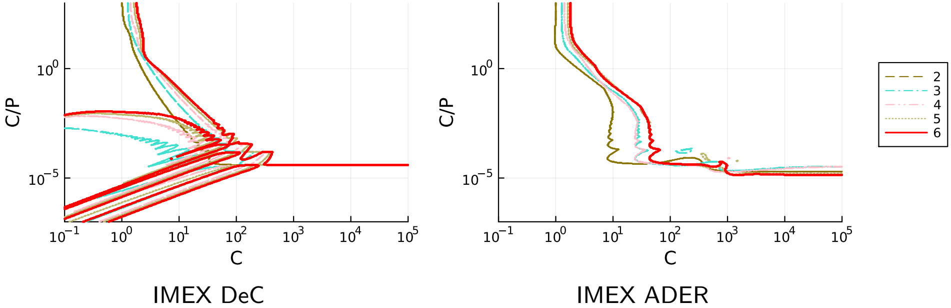


$C-E$ stability plots for IMEX DeC/ADER on advection–diffusion

- Advection $Du_j = \frac{u_j - u_{j-1}}{\Delta x}$ first order
- Dispersion $D_3 u_j =$
  $\frac{1}{4h^3}\left(-u_{j-2} - u_{j-1} + 10u_j - 14u_{j+1} + 7u_{j+2} - u_{j+3}\right).$
  third order
- **Time orders** from 2 to 6

Gauss–Lobatto



Stability areas for orders 2 to 6 with Gauss-Lobatto nodes.

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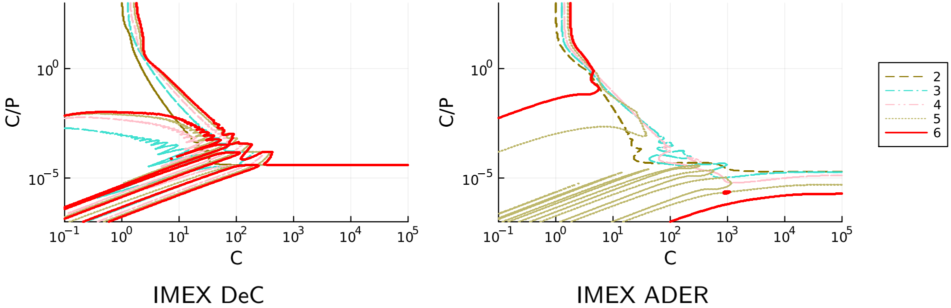

Stability areas for orders 2 to 6 with equispaced nodes.

$C-E$ stability plots for IMEX DeC/ADER on advection–dispersion

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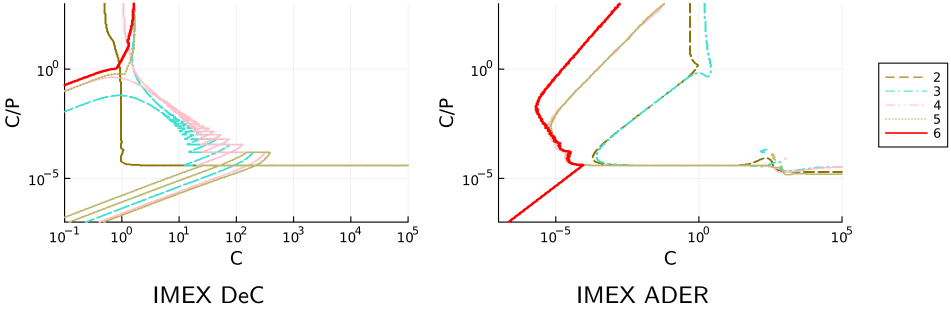
- **Diffusion operator** order k
- **Time order** k from 2 to 6

Gauss–Lobatto



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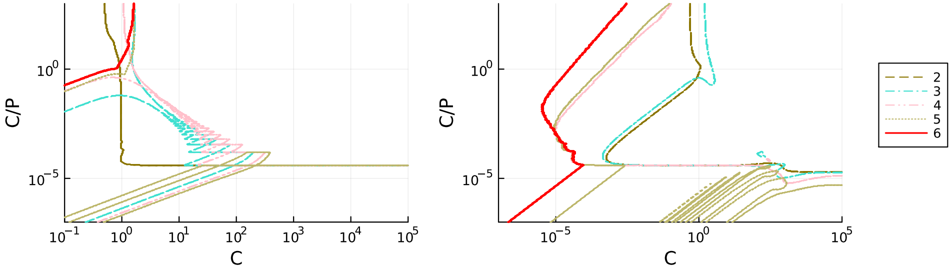

IMEX DeC IMEX ADER

Figure: Stability areas for orders 2 to 6 with equispaced nodes.