A time-adaptive solver for pressure dominated flows in CFD and FSI: domain decomposition and model reduction

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Reduced Order Models for FSI with domain decomposition

Equations

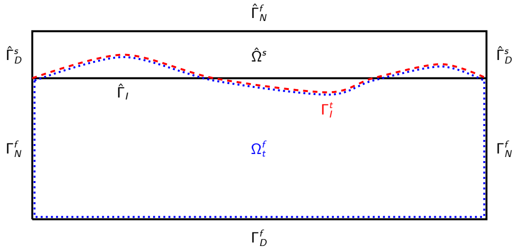
- Fluid

$$\begin{cases} \rho_f \left[\partial_t u_f + (u_f \cdot \nabla)\, u_f\right] - \mathrm{div}\sigma_f(u_f, p_f) = b_f \\ -\mathrm{div} u_f = 0 \end{cases}$$

- Structure

$$\rho_s \partial_t^2 \hat{d}_s - \widehat{\mathrm{div}} \hat{P}(\hat{d}_s) = \hat{b}_s$$

- Interaction
  - Continuity of the velocities u_f and $\frac{\mathrm{d}}{\mathrm{d}t} d_s$
  - Balance of stresses
  - Continuity of displacements (?)
- Boundary/initial conditions



Reduced Order Models for FSI with domain decomposition

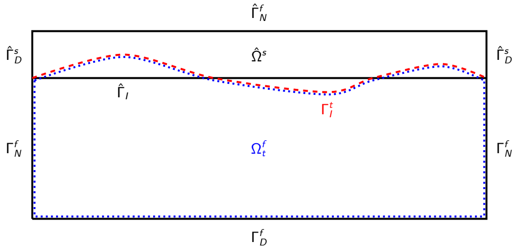
Techniques

- Arbitrary Lagrangian-Eulerian (ALE) map for fluid changing domain

$$\mathcal{A}_t : \hat{\Omega}^f \to \Omega_t^f$$
$$\hat{x} \mapsto x = \hat{x} + \hat{d}_f(\hat{x}, t)$$

- Extension problem $\sigma^{ext}(\hat{d}_f(t)) = 0$
- Interface condition (ALE form) on $\hat{\Gamma}_I$
  - $\hat{u}_f = \frac{\mathrm{d}}{\mathrm{d}t}\hat{d}_s$
  - $\hat{J}\hat{\sigma}_f(\hat{u}_f, \hat{p}_f)\hat{F}^{-T}\hat{\boldsymbol{n}}_f = -\hat{P}(\hat{d}_s)\hat{\boldsymbol{n}}_s$
  - $\hat{d}_f = \hat{d}_s$
- Deformation gradient

$$\hat{F} := \hat{\nabla}\mathcal{A}_t$$
$$\hat{J} := \det \hat{F}$$



Cauchy stress tensor

$$\hat{\sigma}_f(\hat{u}_f, \hat{p}_f) = \hat{\sigma}_f^{du}(\hat{d}_f, \hat{u}_f) + \hat{\sigma}_f^{p}(\hat{p}_f)$$
$$\hat{\sigma}_f^{du}(\hat{d}_f, \hat{u}_f) := \rho_f \nu_f \left(\hat{\nabla}\hat{u}_f \hat{F}^{-1} + \hat{F}^{-T}\hat{\nabla}^T\hat{u}_f\right)$$
$$\hat{\sigma}_f^{p}(\hat{p}_f) := -\hat{p}_f I$$

Monolithic vs Partitioned

Monolithic

- Global functional spaces
- Interface continuity conditions incorporated into the spaces
- Fully coupled problem
- Computationally expensive (sometimes prohibitive)
- In general stable

Domain Decomposition / Partitioned / Segregated

- Local spaces
- Iterative procedure
- Use the state-of-the-art codes for each subcomponent
- Computationally effective
- Subject to possible stability issues

Model order reduction

- Both expensive
- Parametric context too expensive
- Reduce costs with MOR
- Intrusive (POD-Galerkin + hyper-reduction) vs non-intrusive (POD-NN)

Model order reduction offline costs and tests

Computational problems

- Offline we need some FOM simulations
- Both monolithic and DD are anyway expensive
- Simulations can last some days to have reasonable accuracy

Test cases

- CFD test (no structure)
  - Backward facing step
  - Left inlet BC, right outflow, top/bottom walls
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- FSI test
  - Double leaflets in channel (haemodynamics)
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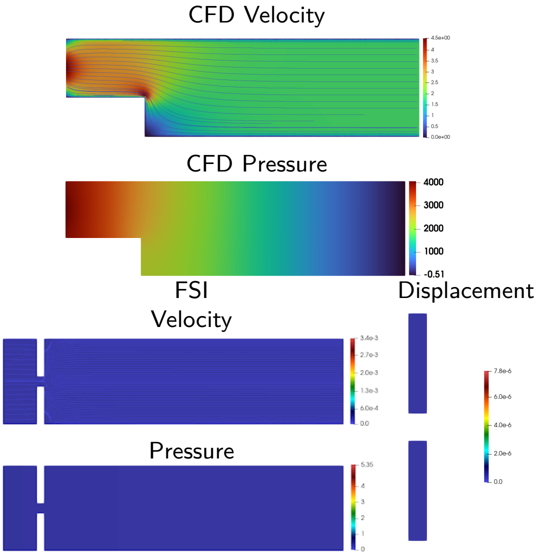
  - $\Delta t = 10^{-4}$ and $T_{end} = 0.01$



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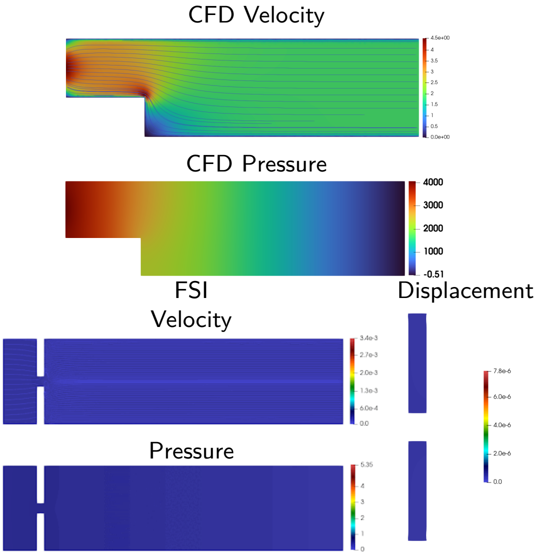

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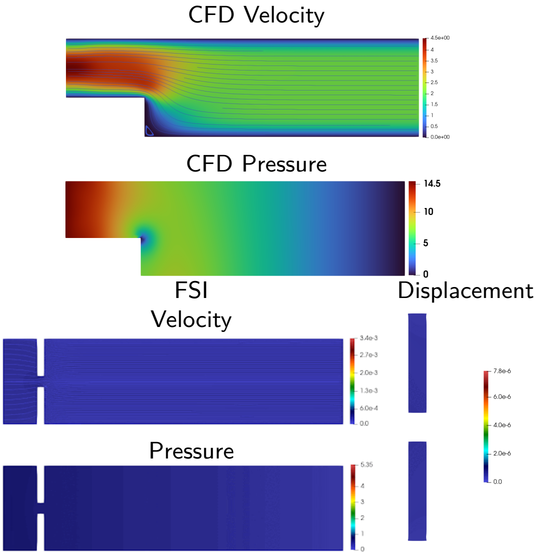

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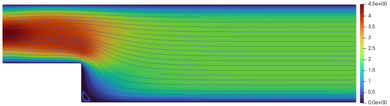
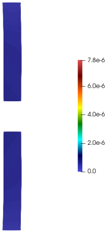


CFD Pressure

FSI Velocity

Displacement



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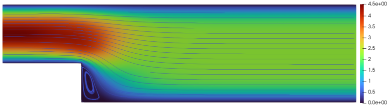

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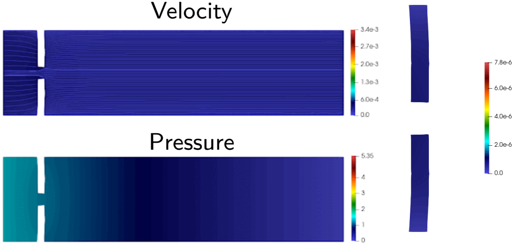

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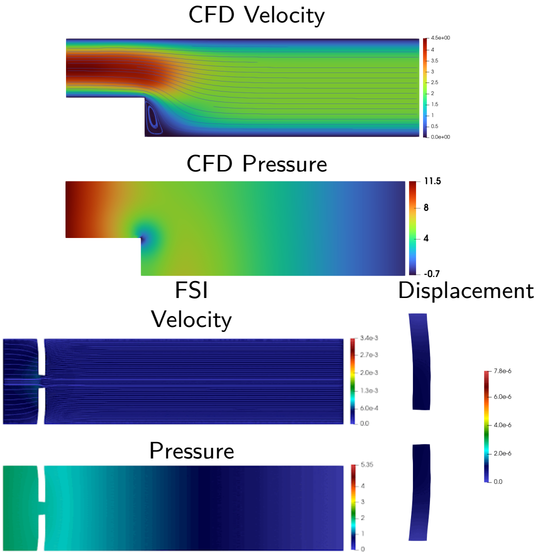

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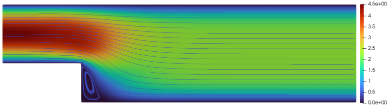

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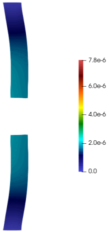

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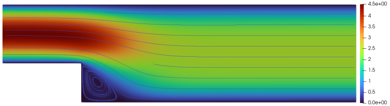

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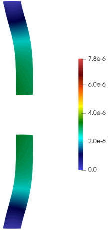

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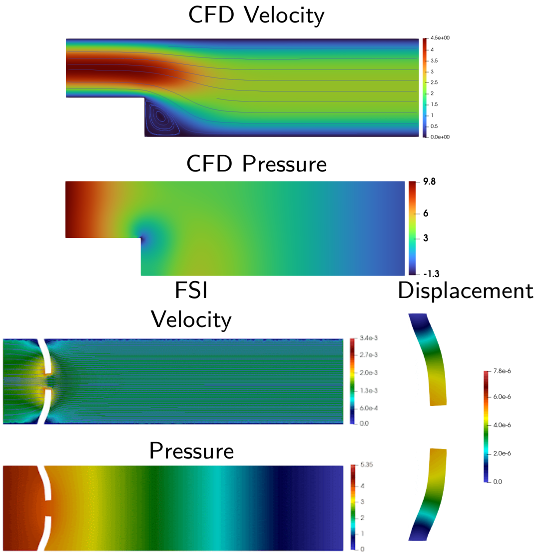

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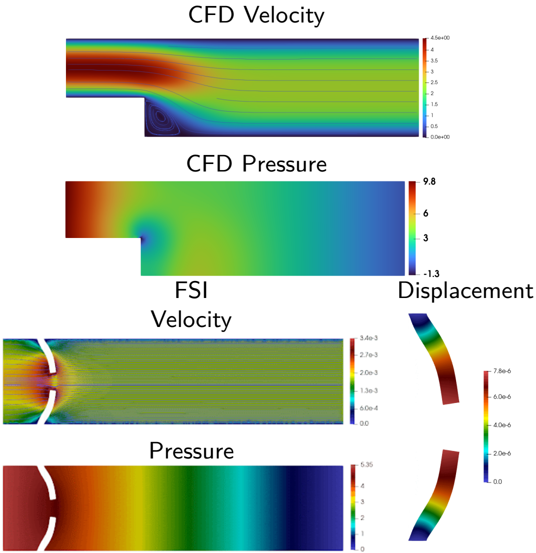

Time adaptivity!

Classical time-adaptivity

- Two time integrators: A, B
- Either order of A different from order of B: $e_A = O(\Delta t^{p_A})$, $e_B = O(\Delta t^{p_B})$ with $p_A < p_B$
- Constant in front of the error is different in A and B (and proportionality is known): $e_A = C_A C \Delta t^p$, $e_B = C_B C \Delta t^p$ with $C_A \neq C_B$
- $e_A - e_B = u_A - u_B \approx e_A$
- Choose Δt according to error and tolerances
  - low error $\Longrightarrow$ increase Δt
  - large error $\Longrightarrow$ decrease Δt
- We will use implicit BDF2 and implicit BDF3
  - BDF2 is stable and cheap
  - BDF3 less stable, but good for estimator
  - Difference is in the way they discretize ∂_t

Saddle-point problems estimators

- ☺ $\partial_t u$ is in the equation
- ☹ $\partial_t p$ is not in the equation
- Many estimators consider only u

Time adaptivity!

Our saddle-point estimator

- $\max_{\xi\in\{u,p,d\}} \|\xi^{n+1,\text{BDF2}} - \xi^{n+1,\text{BDF3}}\|_{L^2(\Omega)}$
- Cheaper way to compute ξ^{BDF3} still being 3rd order
- find $\tilde{\boldsymbol{U}}_h^{n+1,\text{BDF3}} = \boldsymbol{U}_h^{n+1,\text{BDF2}} + \delta \boldsymbol{U}_h^{n+1,\text{BDF3}} \in W$, where $\delta \boldsymbol{U}_h^{n+1,\text{BDF3}} \in W_0$ is the solution of the following linearised equation

$$J\left[R\left(\Xi_{n+1}^{\text{BDF3}}\left(\boldsymbol{U}_h^{\text{BDF2}}\right), \boldsymbol{U}_h^{n+1,\text{BDF2}}\right)\right]\left(\delta \boldsymbol{U}_h^{n+1,\text{BDF3}}\right) = -R\left(\Xi_{n+1}^{\text{BDF3}}\left(\boldsymbol{U}_h^{\text{BDF2}}\right), \boldsymbol{U}_h^{n+1,\text{BDF2}}\right)$$

- Nonlinear solver to get $\boldsymbol{U}_h^{n+1,\text{BDF2}}$

☺ Linear solver to get $\tilde{\boldsymbol{U}}_h^{n+1,\text{BDF3}}$

Numerical simulations: CFD

Backward-facing step flow CFD

- Incompressible Navier-Stokes
- $\mathbb{P}^2 - \mathbb{P}^1$ Taylor Hood FEM spaces
- Inflow $u_{in} = (\varphi(t)\frac{20}{9}(y-2)(5-y), 0)$ on Γ_{in}
- $\varphi(t) = 0.5(1-\cos(\pi t))$ for $t \leq 1$, then $\varphi(t) = 1$
- Outflow homogeneous Neumann for all variables
- No slip wall on Γ_{wall}, i.e., Dirichlet homogeneous on u
- $\nu = 0.05$ with Reynolds number 300

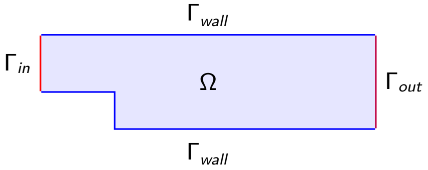

Figure: Backward-facing step domain

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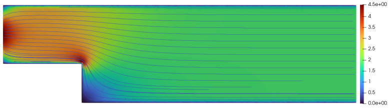

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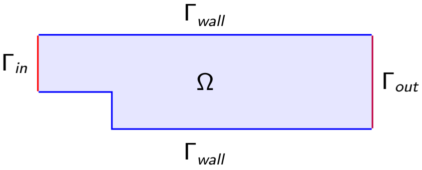

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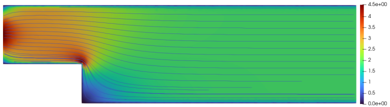

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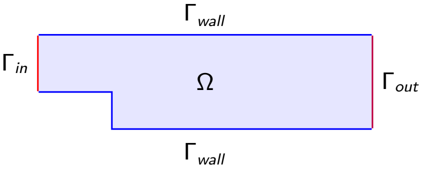

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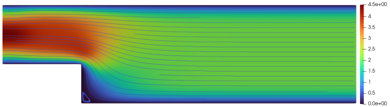

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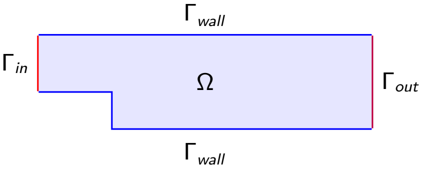

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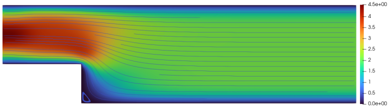

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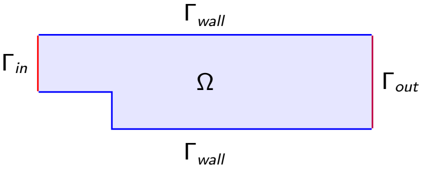

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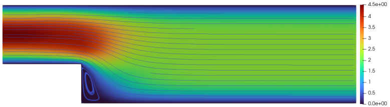

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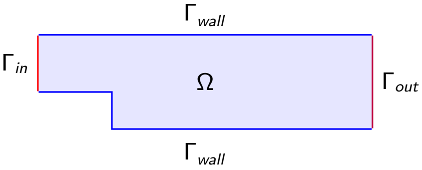

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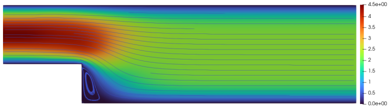

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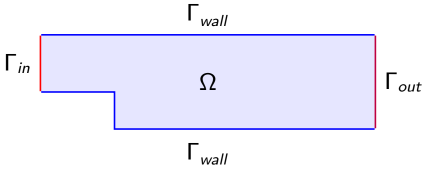

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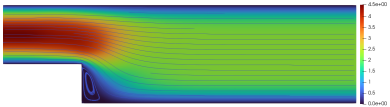

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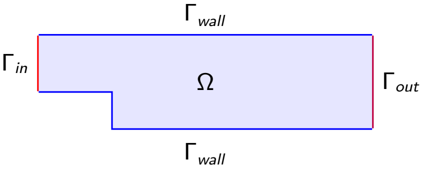

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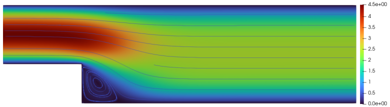
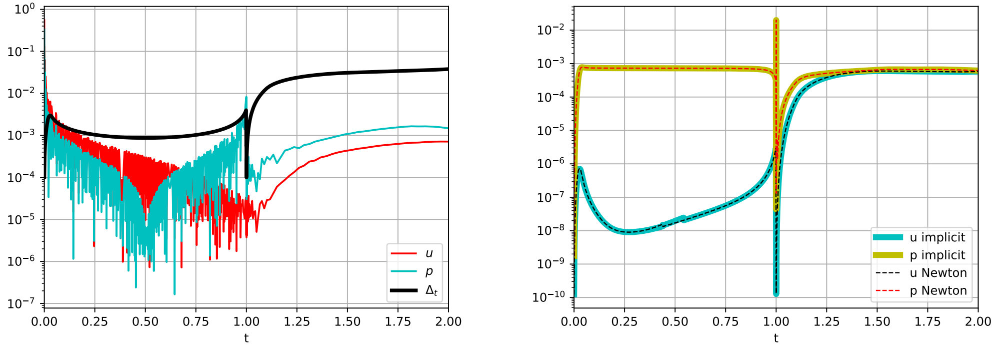


CFD Pressure

Numerical simulations: CFD



(a) **CFD-300**: errors and timestep (b) **CFD-300**: error estimators

Figure: Adaptive time-steps distribution and relative errors w.r.t. constant-timestep solution (left) and the comparison of implicit and linear implicit (Newton) time estimators (right)

Numerical simulation: FSI monolithic

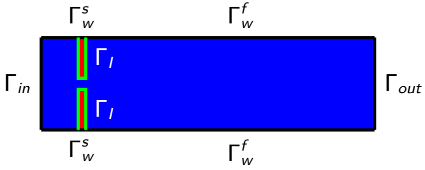
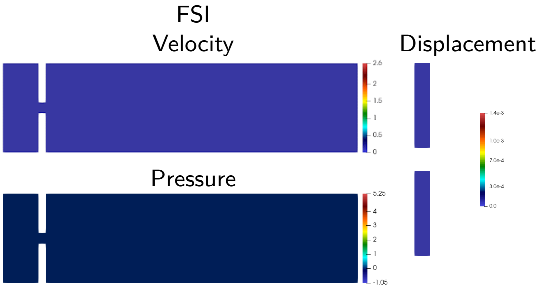


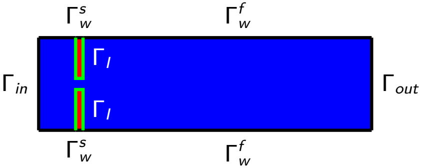
Figure: Reference FSI domain

Heamodynamic problem

- $\nu_f = 0.035, \rho_f = 1, \rho_s = 1.1$
- $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_2$ with $67,390$ DoFs
- $p_{in}(t) = \begin{cases} 5\left(1-\cos(\frac{\pi t}{0.2})\right) & \text{for } t \leq 0.1, \\ 5 & \text{for } t > 0.1. \end{cases}$
  through a Neumann condition in Γ_{in}
  $u_N^f(x,t) = -p_{in}(t)\mathbf{n_f}(\mathbf{x})$

☺ $T_{end} = 2$





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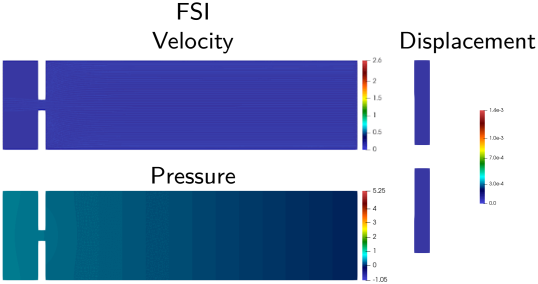
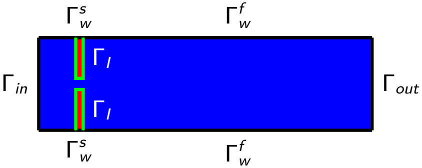


Numerical simulation: FSI monolithic



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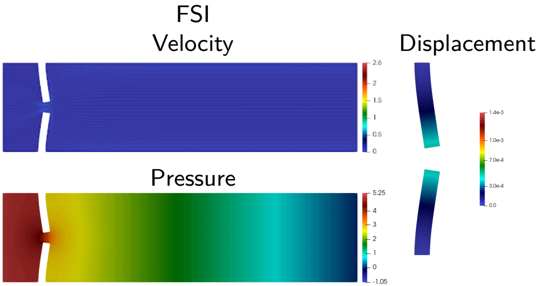

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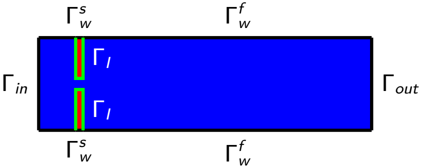

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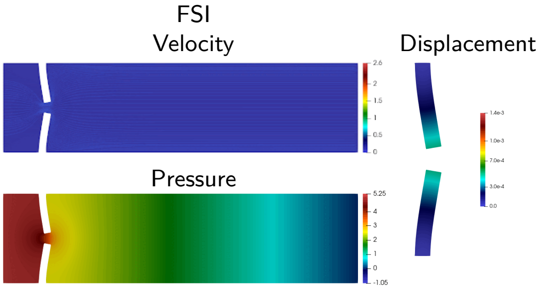

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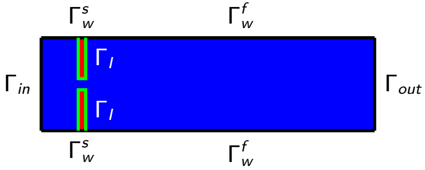

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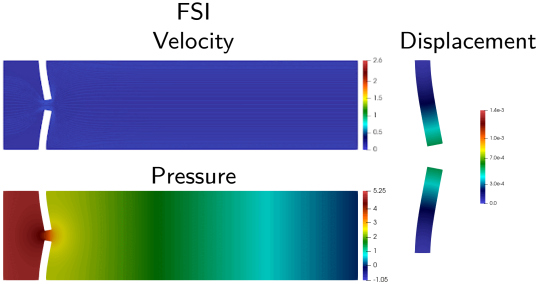

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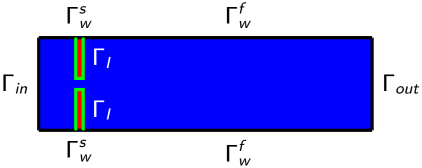

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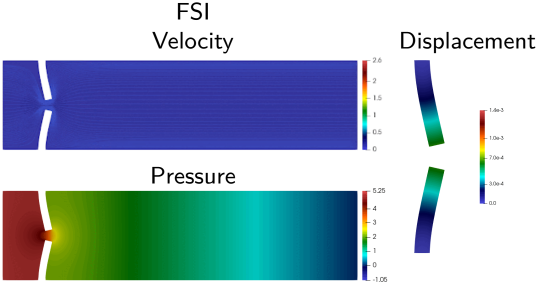

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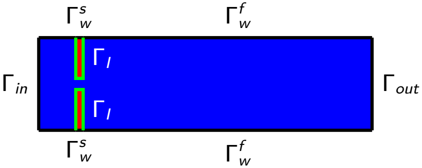

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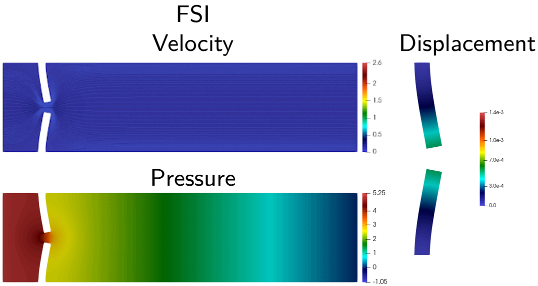
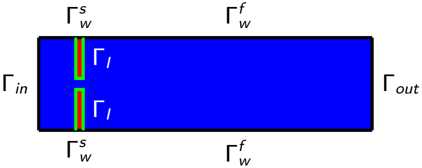


Numerical simulation: FSI monolithic



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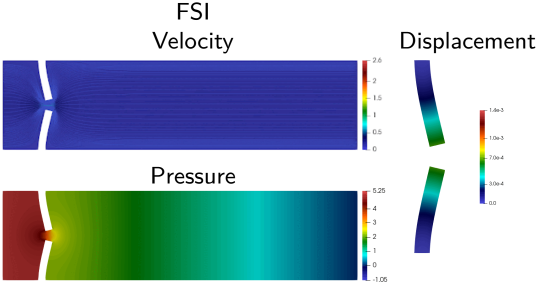

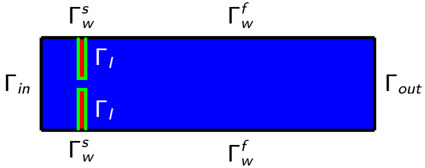

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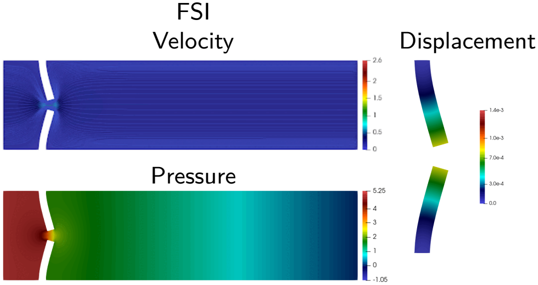

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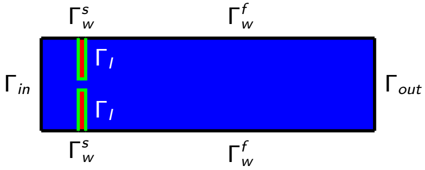

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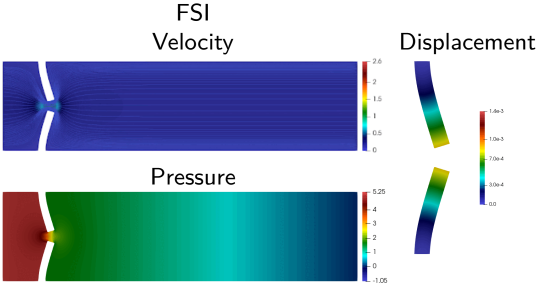

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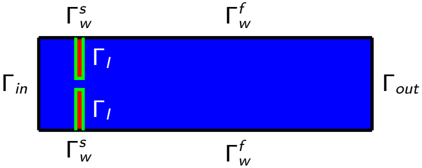

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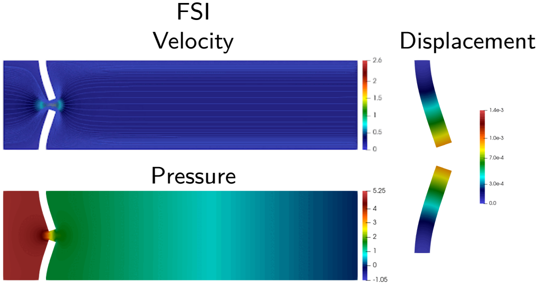

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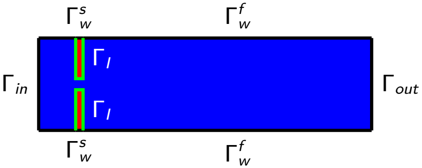

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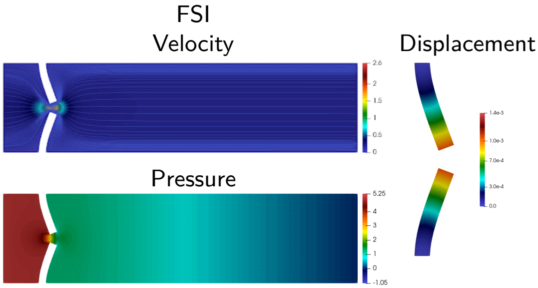

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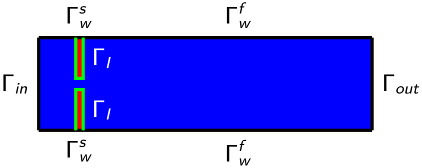

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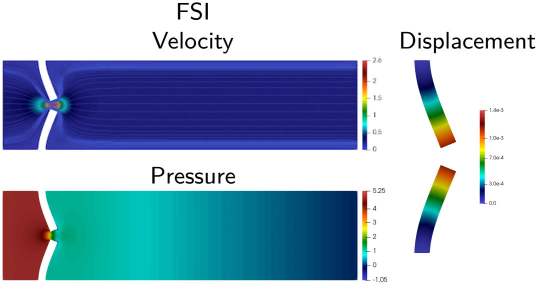

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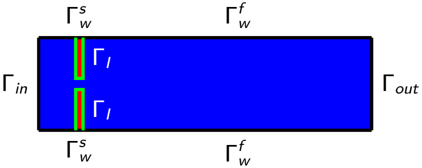

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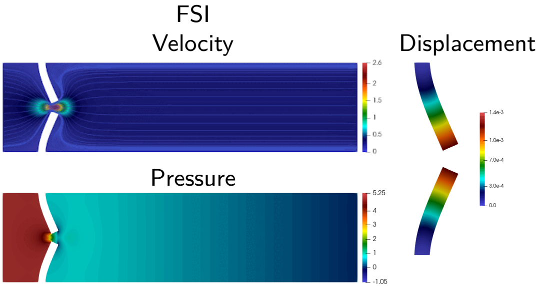
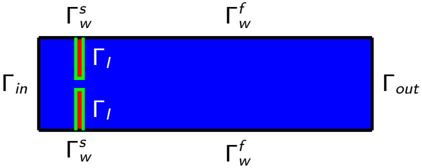


Numerical simulation: FSI monolithic



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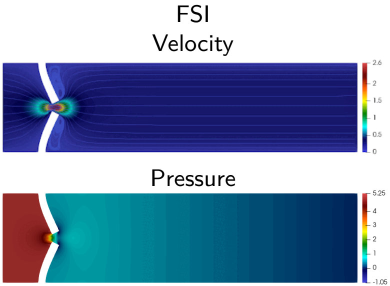

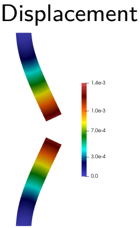

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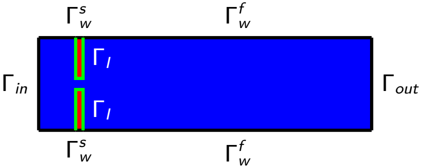

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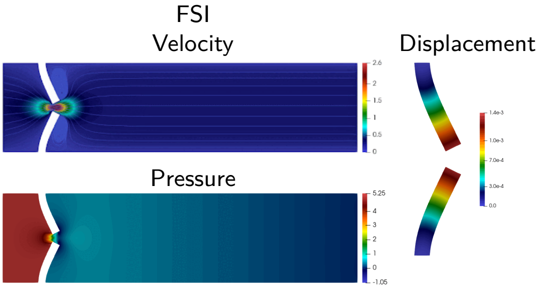

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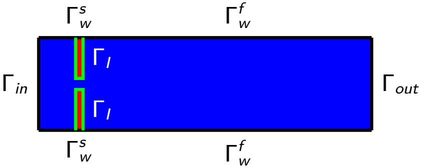

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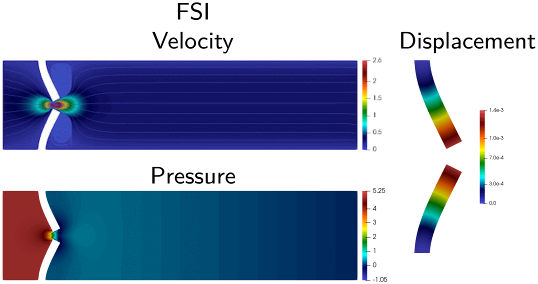
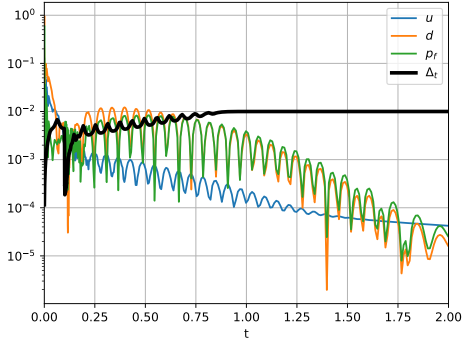
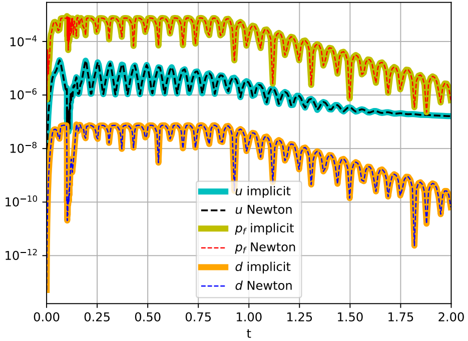


Numerical simulations: FSI monolithic approach



(a) **FSI-H2**: errors and timestep



(b) **FSI-H2**: error estimators

Figure: Adaptive time-steps distribution and relative errors w.r.t. constant-timestep solution (left) and the comparison of implicit and LI time estimators (right)

Computational costs

Table: Computational cost comparison between constant BDF2 and time adaptive algorithms: total simulation time (left) and time of one evaluation of the error estimator (mean $\pm$ std)

	Computational time		Estimator cost		Timesteps number		
Test	Constant	LI adaptive	Implicit	LI	Const.	Impl.	LI
CFD	12 hours	2 hours	2.1 ± 0.2 sec	1.68 ± 0.1 sec	20,000	976	976
FSI	132 hours	5 hours	37 ± 4 sec	25 ± 2 sec	20,000	347	360

FSI + DD

Variational spaces

- $u_f \in V_f \subset H^1(\Omega^f)^2$
- $p_f \in Q_f = L^2(\Omega^f)$
- $d_f \in E_f \subset H^1_D(\Omega^f)^2$
- $d_s \in E_s \subset H^1_D(\Omega^s)^2$
- $u_s \in V_s \subset H^1_D(\Omega^s)^2$
- $g_I \in V_I \subset H^1(\Gamma_I)^2$

Interface variable

$$g_I := -P(d_s)\mathbf{n}_s = J\sigma_f^{du}(d_f, u_f)F^{-T}\mathbf{n}_f + J\sigma_f^{p}(p_f)F^{-T}\mathbf{n}_f$$

Fluid

$$\begin{aligned} m_f(\partial_t u_f, v_f; d_f) + a_f(u_f, v_f; d_f) + c_f^{ALE}(\partial_t d_f, v_f, u_f; d_f) \\ + b_f^A(p_f, v_f; d_f) + c_f(u_f, u_f, v_f; d_f) \\ = f_f(v_f; d_f) + \left(u_N^f, v_f\right)_{\Gamma_N^f} + (g_I, v_f)_{\Gamma_I} \quad \forall v_f \in V_f, \end{aligned}$$

$$b_f^B(u_f, q_f; d_f) = 0 \qquad \forall q_f \in Q_f,$$

$$a_f^e(d_f, e_f) = 0, \quad \forall e_f \in E_f$$

$$d_f = d_s \text{ on } \Gamma_I.$$

Structure

$$m_s(\partial_t u_s, v_s) + a_s(d_s, v_s) = f_s(v_s) + (d_N^s, v_s)_{\Gamma_N^s} - (g_I, v_s)_{\Gamma_I} \quad \forall v_s \in V_s,$$

$$(\partial_t d_s, e_s)_{\Omega_s} = (u_s, e_s)_{\Omega_s} \quad \forall e_s \in E_s.$$

Objective Functional

$$J_\gamma(u_f, u_s; g_I) := \frac{1}{2}\int_{\Gamma_I} |u_f - u_s|^2 \mathrm{d}\Gamma + \frac{\gamma}{2}\int_{\Gamma_I} |g_I|^2 \mathrm{d}\Gamma$$

FSI + DD: Optimization strategy for Nonlinear Least Squares problems

Line search methods

- ☹ Not too good for our applications
- ☹ Gauss-Newton is “too local” - diverges after the first 12 time steps
- ☹ Finds infeasible directions for the nonlinear solvers
- ☹ Gradient-based methods stagnates too much and are not always able to overcome “flat” areas

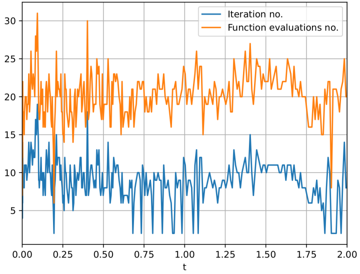
Trust region methods

- Compute (usually) quadratic approximation of the objective functional around the iterative point
- Update trust region radius depending on how well the model approximates the objective
- Solve model optimization problem within the trust region
- Subspace Interior Trust Region Method (STIR)[a]

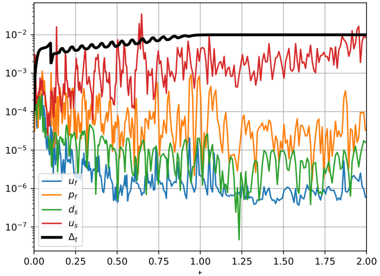
[a] M. A. Branch et al. A Subspace, Interior, and Conjugate Gradient Method for Large-Scale Bound-Constrained Minimization Problems, 1999

Simulations: Haemodynamics FSI DD STIR time-adaptive

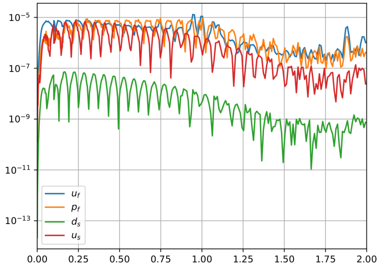
Time adaptive $\Longrightarrow$ 302 timesteps
Iterations and **functional evaluations** (only 1 gradient evaluation per iteration)



Relative errors
Reference solution: monolithic with $\Delta t = 10^{-4}$



Estimators with linear implicit BDF2-BDF3 approach



Perspectives

ROM

- Intrusive
  - ☺ Galerkin Projection (some effort for Jacobian)
  - ☹ Hyper-reduction
- Nonintrusive: POD-NN
  - ☺ No further models
  - ☹ Exponential behaviors are difficult for NN
- Other questions
  - Time-adaptivity for ROM is necessary or a waste of time?

Bibliography

- Prusak, Nonino, Torlo, Ballarin and Rozza. An optimisation-based domain-decomposition reduced order model for the incompressible Navier-Stokes equations, CAMWA, 2023
- Prusak, Torlo, Nonino and Rozza. An optimisation-based domain-decomposition reduced order model for parameter-dependent non-stationary fluid dynamics problems, CAMWA, 2024
- I. Prusak. Application of optimisation-based domain-decomposition reduced order models to parameter-dependent fluid dynamics and multiphysics problems, PhD Thesis, 2023
- Prusak, Torlo, Nonino and Rozza. A time-adaptive algorithm for pressure dominated flows: a heuristic estimator, 2025

Perspectives

- More complex domains (multi-domain decomposition)
- Dimensionality reduction of the Jacobian
- Matrix-free methods
- Optimal control problems or inverse problems
- Heterogeneous coupling (FOM-ROM)

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ROM

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#ADV

Summer School in Rome, September 15-19

Numerical methods for high-dimensional data

Lecturers: Despres, Herty, Karniadakis

THANKS!!

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