How to Preserve Moving Equilibria: Global Flux and Analytical Methods

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SAPIENZA
UNIVERSITÀ DI ROMA

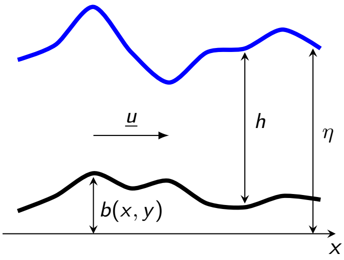
Water equilibria and perturbations

- Lake at rest perturbation
- Moving stationary wave
- Vortex type stationary solutions

Equilibria for shallow water equations

Shallow Water Equations

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y\left(huv\right) = -gh\partial_x b \\ \partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = -gh\partial_y b \end{cases}$$



Equilibria for shallow water equations

Shallow Water Equations

$$
\begin{cases}
\partial_t h + \partial_x(hu) + \partial_y(hv) = 0\\
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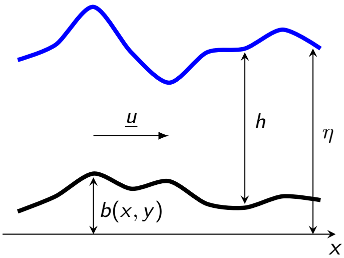
\end{cases}
$$

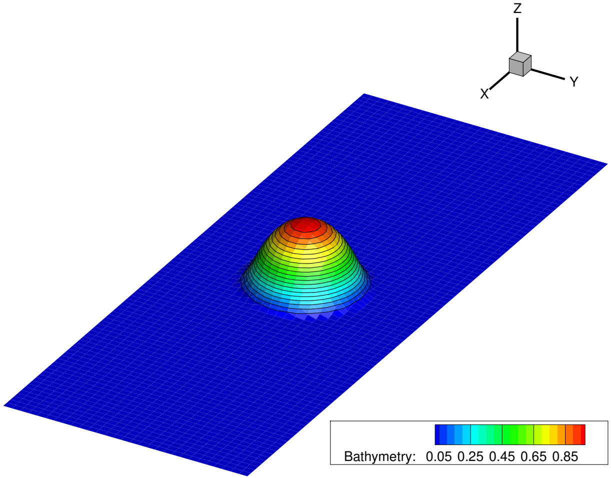
Lake at rest equilibrium

$$
h(x,y) + b(x,y) \equiv \eta_0 \qquad u(x,y) = v(x,y) \equiv 0
$$

$$
\partial_x\left(\frac{g}{2}h^2\right) + gh\partial_x b = gh\partial_x h + gh\partial_x b = gh\partial_x \eta_0 = 0.
$$



Simulation example lake at rest with perturbation



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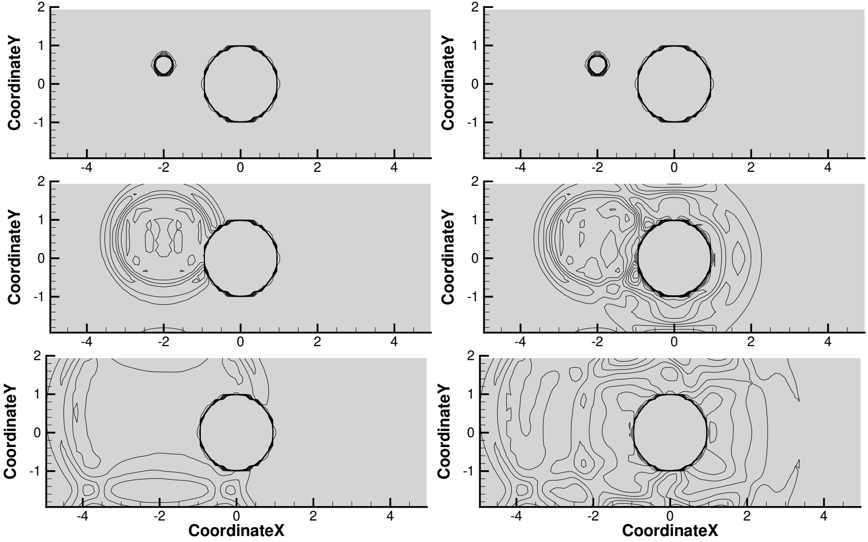

Equilibria for shallow water equations

Shallow Water Equations 1D

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = -gh\partial_x b \end{cases}$$

Stationary waves in 1D

$$hu(x) =: q(x) \equiv q_0^x$$

and h such that

$$\partial_x \left(hu^2 + \frac{g}{2} h^2 \right) + gh\partial_x b = 0$$

$$\dots$$

$$\partial_x \left(\frac{q^2}{2gh^2} + h + b \right) = 0$$

$$\frac{q^2}{2gh^2(x)} + h(x) + b(x) = \mathcal{Q}(x_0) \qquad (1)$$

Equilibria for shallow water equations

Shallow Water Equations 1D

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Cubic equation solutions

- Supercritical state $u > \sqrt{gh}$
- Subcritical state $u < \sqrt{gh}$
- Negative h

Stationary waves in 1D

$$hu(x) =: q(x) \equiv q_0^x$$

and h such that

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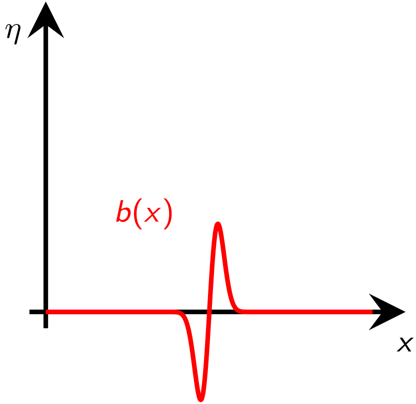
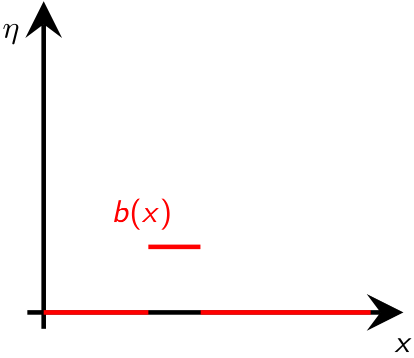
Simulation example moving equilibria non flat bathymetry

Continuous Bathymetry



Discontinuous Bathymetry



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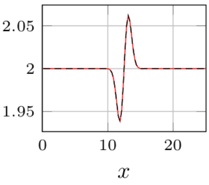
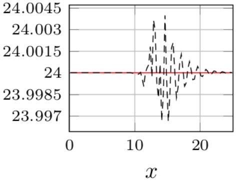
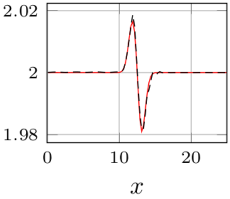


(a) η



(b) q



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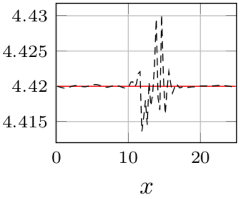
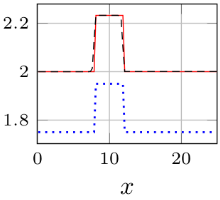


(b) q

Discontinuous Bathymetry



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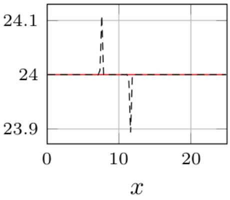
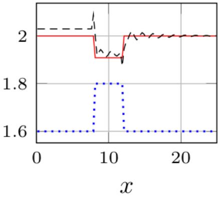


(b) q



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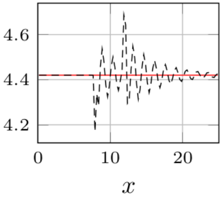


(b) q

2

Equilibria for shallow water equations

Shallow Water Equations (no bathymetry)

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y\left(huv\right) = 0 \\ \partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = 0 \end{cases}$$

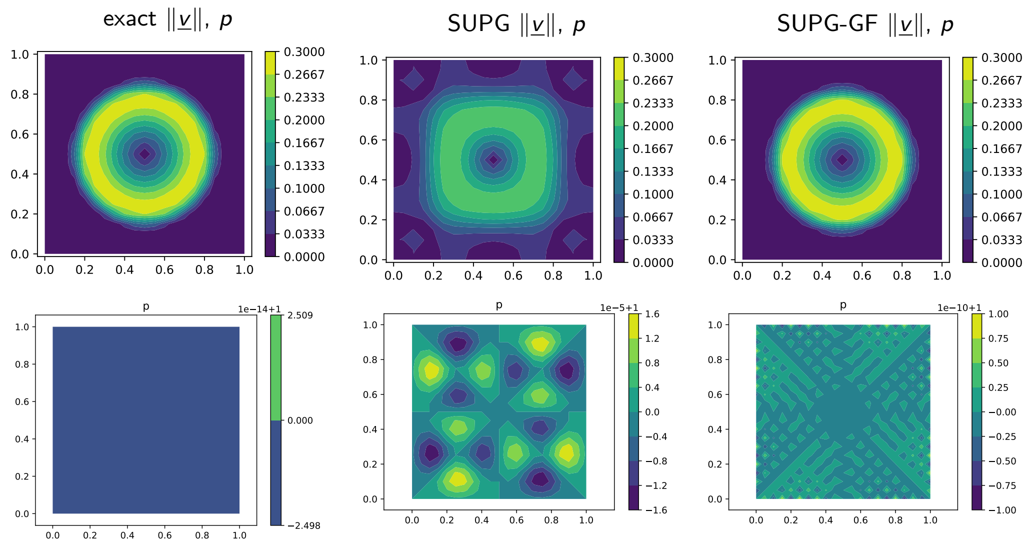
Vortices: Div-free solutions

$$\begin{cases} r = (x-x_0)^2 + (y-y_0)^2 \quad \theta = \arctan\left(\frac{y-y_0}{x-x_0}\right) \\ u(r) = -\sin(\theta)u_\theta(r) \qquad v(r) = \cos(\theta)u_\theta(r) \\ h(r): \quad h'(r)gr = u_\theta^2(r) \end{cases}$$

Other equations

- Euler equations (isentropic)
- Linear Acoustic equations

Simulation example of a vortex (for linear acoustics)



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How can we preserve the equilibria?

Exactly!

Impossible: discretization of data b, of the solutions h, u, v

Exactly with respect to discretization

- Possible
- Might involve some analytical equation to be solved
- Requires the knowledge a priori of equilibria type

Exactly Well Balancing

Better than the underlying method

- Possible
- No need of inverting analytical equations
- No need of a priori knowledge of the equilibrium type

Well Balancing

State of the art techniques (part 1)

Subtract equilibrium

- Know analytical equilibrium
- Dedner 2004[a] and Berberich 2021[b]

[a] Dedner, A., Rohde, C., Schupp, B., & Wesenberg, M. (2004). Computing and Visualization in Science, 7(2), 79-96.

[b] J. P. Berberich, P. Chandrashekar, and C. Klingenberg. Computers & Fluids, 219:104858, 2021.

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq} := (h^{eq}, u^{eq}, v^{eq})$
- Discrete exact equilibrium residual: $\mathcal{S}^{eq}(t^n) := \mathcal{S}(V^{eq}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq}(t^n)$

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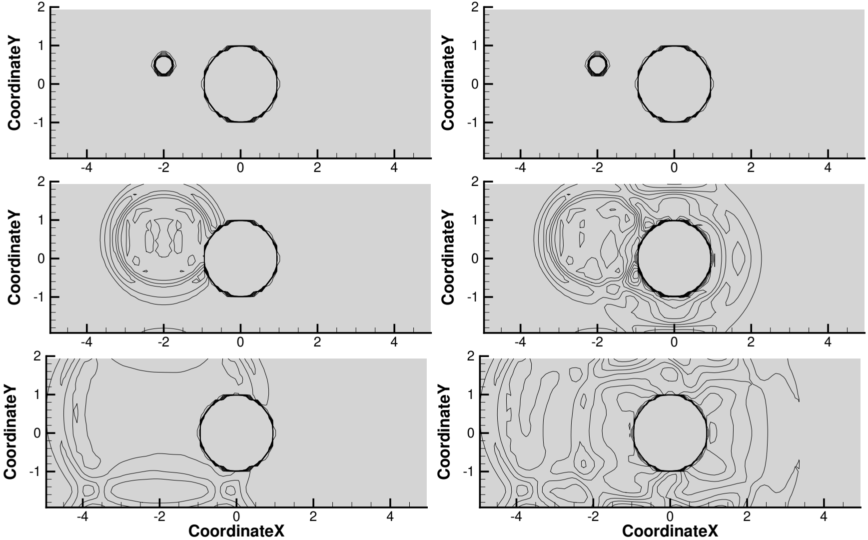
Properties

- ☺ Ridiculously well balanced: $V^n = V^{eq} \Longrightarrow V^{n+1} = V^{eq}$
- ☹ Know equlibrium a priori
- ☺ Lake at rest
- ☹ Stationary waves
- ☹ 2D vortices

Example: subtract equilibrium[1]



[1] Ciallella, M., Micalizzi, L., Öffner, P., & Torlo, D. (2022). Computers & Fluids, 247, 105630.

State of the art techniques (part 2)[2]

Equilibrium reconstruction

- In every cell solve an ODE at reconstruction/quadrature points, constrained with the state V^n (BVP)
- ODE solver either exact or very accurate
- Malaga school

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq,ODE}$:=ODE_Solver(1) subject to V^n
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[2] Castro, M. J., & Parés, C. (2020). Journal of Scientific Computing, 82(2), 48.

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Properties

☺ Exactly well-balanced $V^n = V^{eq,ODE} \Longrightarrow V^{n+1} = V^{eq,ODE}$

☺ For all equilibria of one type

☹ Expensive (ODE solver for each cell)

☺ Lake at rest

☺ Stationary waves

☹ Problem for transcritical flows $u = \sqrt{gh}$

☹ 2D vortices

[2] Castro, M. J., & Parés, C. (2020). Journal of Scientific Computing, 82(2), 48.

State of the art techniques (part 3)[3]

Riemann problem modification

- For FV schemes
- Change the Riemann problem approximation
- Exploit (1) such that at equilibrium it is satisfied by the Riemann problem
- Michel-Dansac 2016

Properties

- Exactly well-balanced (if (1) analytically invertible else accurate solver) $V^n = V^{eq,ODE} \implies V^{n+1} = V^{eq,ODE}$

☺ For all equilibria of one type

☹ Computations by hand for Riemann Solver

☹ Only 1st order, blending with high order

☺ Lake at rest

☺ Stationary waves

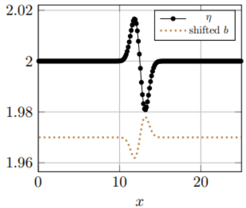
☹ Problem for transcritical flows $u = \sqrt{gh}$

☹ 2D vortices

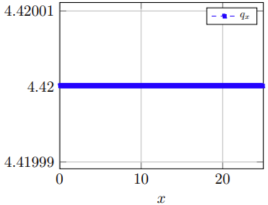
[3] Michel-Dansac, V., Berthon, C., Clain, S., & Foucher, F. (2016). Computers & Mathematics with Applications, 72(3), 568-593.

Example: Riemann Problem Change[4]

SUBCRITICAL

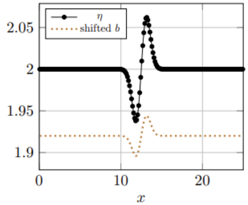


(a) free surface η and bathymetry b, shifted and rescaled



(b) discharge q_x

SUPERCRITICAL



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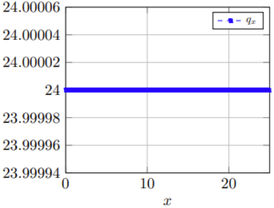

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[4] Ciallella, M., Micalizzi, L., Michel-Dansac, V., Öffner, P., & Torlo, D. (2024). arXiv preprint arXiv:2402.12248.

Example: Riemann Problem Change[4]

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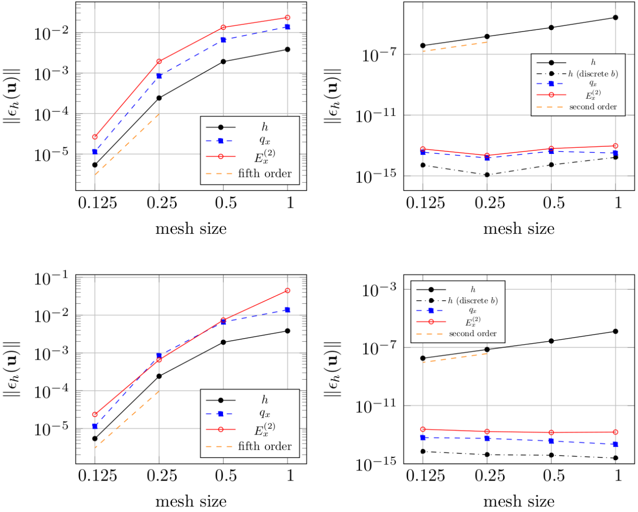

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State of the art techniques (part 4)

Global Flux

- Obtain 1 differential operator for everything
- Put together flux and source
- Integrate the forms
- Gascón 2001[a], Chertock 2022[b], Ciallella 2023[c], Barsukow 2024[d]

[a] Gascón, L., Corberán, J. J. Comput. Phys. 172(1), 261–297 (2001)

[b] Chertock, A., Kurganov, A., Liu, X., Liu, Y., & Wu, T. (2022). Journal of Scientific Computing, 90, 1-21.

[c] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

[d] Barsukow, W., Ricchiuto, M., & Torlo, D. (2024). arXiv preprint arXiv:2407.10579.

1D source recipe

$$\partial_t V + \partial_x f(V) = S(V,x)$$
$$\partial_t V + \partial_x (f(V) - K(V,x)) = 0$$
$$K(V,x) := \int_{x_0}^{x} S(V(s),s) ds$$

2D divergence recipe

$$\partial_t h + \partial_x f + \partial_y g = 0, \qquad f = hu, \quad g = hv,$$
$$\partial_t h + \partial_{xy}(F+G) = 0$$
$$F(x,y) := \int_{y_0}^{y} f(x,\xi) d\xi, \quad G(x,y) := \int_{x_0}^{x} g(\xi,y) d\xi.$$

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Properties

- ☹ Well balanced (not exactly)
- ☺ No need for any analytical equilibria
- ☺ No need for analytical relation
- ☺ No further ODE solver
- ☺ No problems with transcritical points
- ☺ Explicit methods
- ☺ Lake at rest
- ☺ Stationary waves
- ☺ 2D vortices
- ☺ Applicable to FV, FEM, DG

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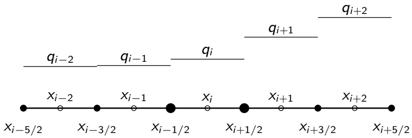
Global Flux in 1D for FV 1st order

$$\partial_t q + \partial_x f(q) = S(q,x) \qquad \Longrightarrow \qquad \partial_t q + \partial_x G(q,x) = 0$$

$$\text{with } G(q,x) := f(q) - K(q,x) = f(q) - \int_{x_0}^{x} S(q(s),s)\mathrm{d}s.$$

$$\text{FV: } q_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\mathrm{d}x$$



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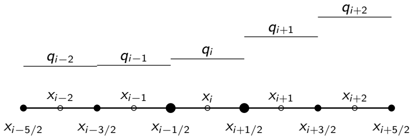

$$f_i := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\mathrm{d}x,$$

$$K_i \approx K(x_i, q(x_i)) = \int_{x_0}^{x_i} S(q(s),s)\mathrm{d}s \approx K_{i-1} + \int_{x_{i-1}}^{x_i} S(q(s),s)\mathrm{d}s,$$

$$K_i := K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}$$

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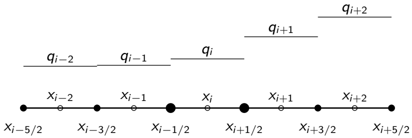

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Numerical flux depends only on G: upwind, Roe, NO Rusanov

$$\partial_t q_i + \frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} = 0,$$

$$\hat{G}_{i+1/2} := \mathrm{sign}(J)^+ G_i + \mathrm{sign}(J)^- G_{i+1},$$

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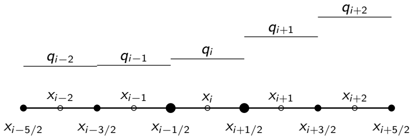

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$$\hat{G}_{i+1/2} := \mathrm{sign}(J)^+ G_i + \mathrm{sign}(J)^- G_{i+1},$$

Equilibrium: $\hat{G}_{i+1/2} = \hat{G}_{i-1/2} = \hat{G}_0$ for all i

$$f_i - K_i = G_0$$

Mind: high order, other equilibria (LAR), super convergence

Developing GF 1D FV 1st order

I want you to hate me, let's do the computations in a simple case (upwind)!

Formulae

- $\partial_t q_i = -\frac{\hat{G}_{i+1/2}-\hat{G}_{i-1/2}}{\Delta x}$
- $G_i = f_i - K_i$
- $K_i = K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}$
- $\operatorname{sign}(J) = +1$
- $\hat{G}_{i+1/2} = G_i$

Classical Upwind FV

$$\partial_t q_i = -\frac{f_i - f_{i-1}}{\Delta x} + S_i$$

Expand!

$$\begin{aligned}
\partial_t q_i &= -\frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} \\
&= -\frac{G_i - G_{i-1}}{\Delta x} \\
&= -\frac{f_i - f_{i-1}}{\Delta x} + \frac{K_i - K_{i-1}}{\Delta x} \\
&= -\frac{f_i - f_{i-1}}{\Delta x} + \frac{S_{i-1} + S_i}{2}.
\end{aligned}$$

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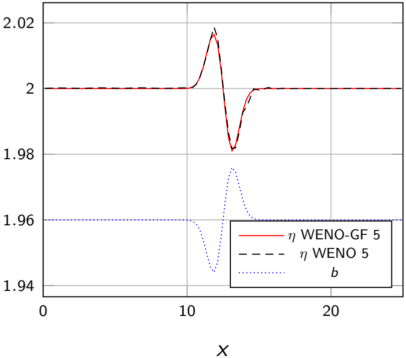
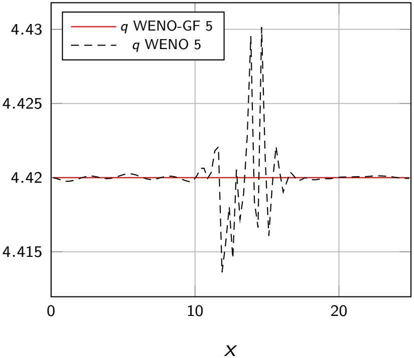
Validation: Subcritical flow and perturbation



(a) η



(b) q

Figure: Subcritical flow: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes with $N_e = 100$.

Validation: Subcritical flow and perturbation

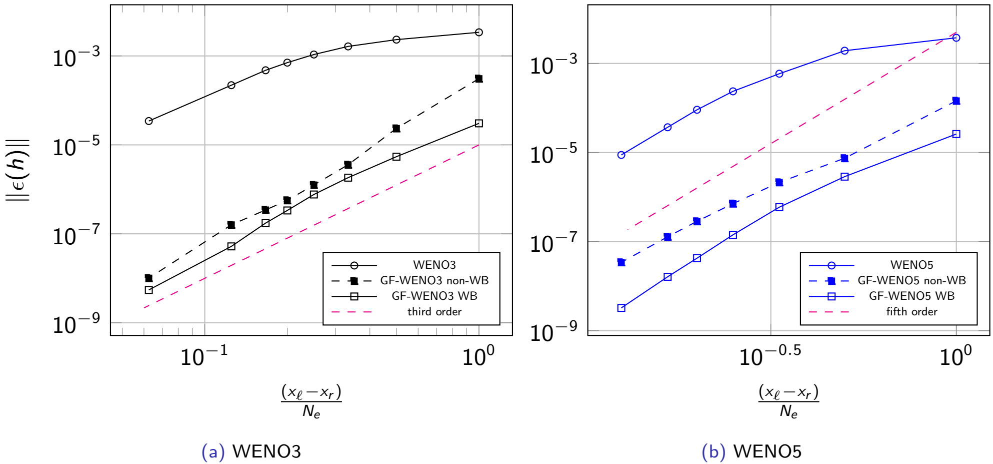
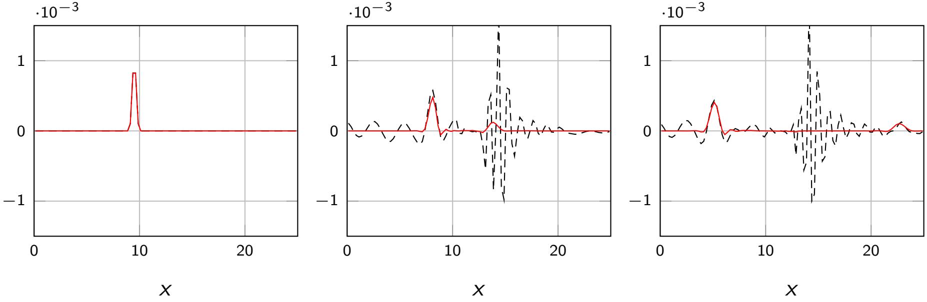


Figure: Subcritical flow: convergence tests with WENO3 and WENO5.

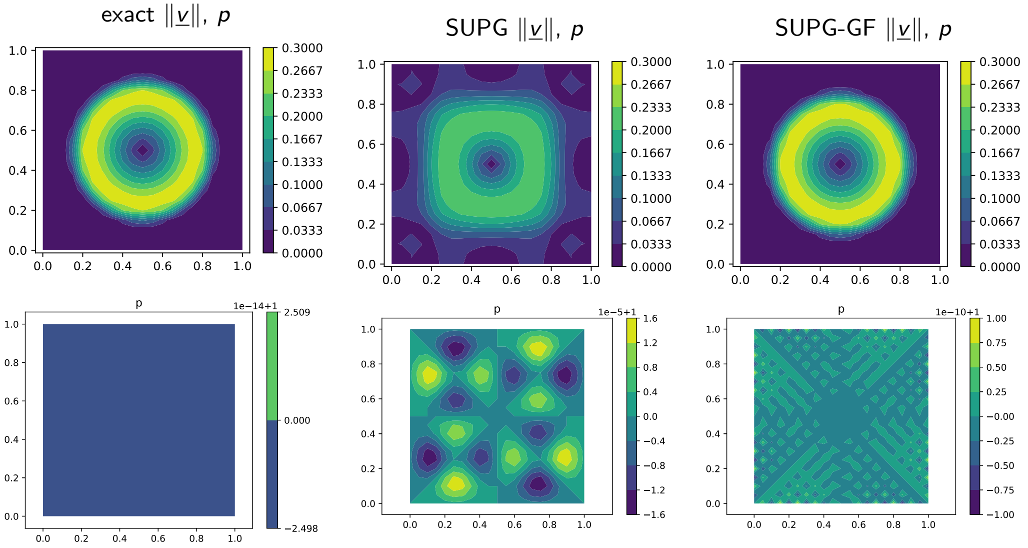
Validation: Subcritical flow and perturbation



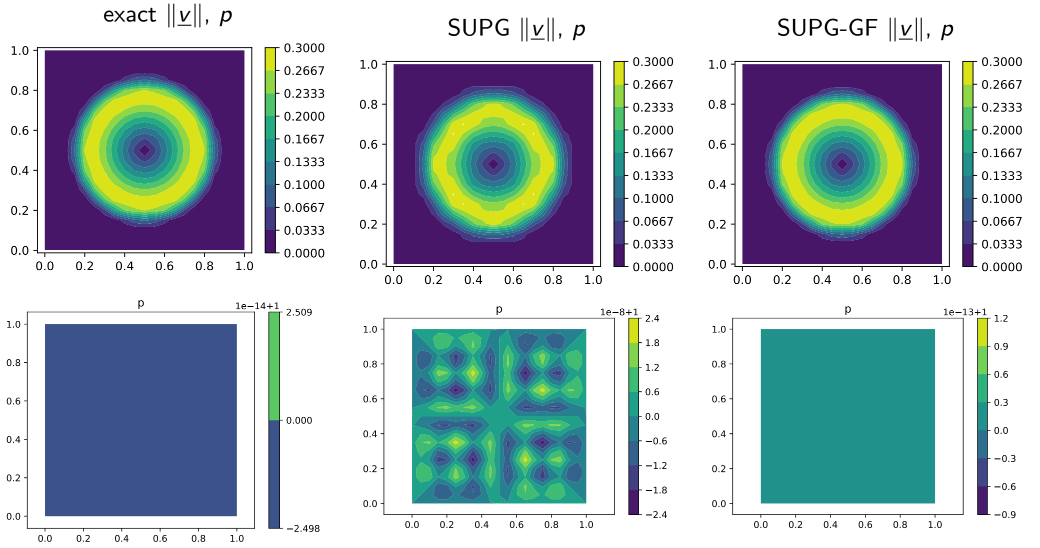
(a) $t = 0$ (b) $t = 0.66$ (c) $t = 2$

Figure: Perturbation on a subcritical flow: $\eta - \eta^{eq}$

Simulation of vortex (linear acoustics) FEM+SUPG: $\mathbb{Q}^1$, $N_x = N_y = 20$



Simulation of vortex (linear acoustics) FEM+SUPG: $\mathbb{Q}^2$, $N_x = N_y = 10$



Simulation of vortex: errors

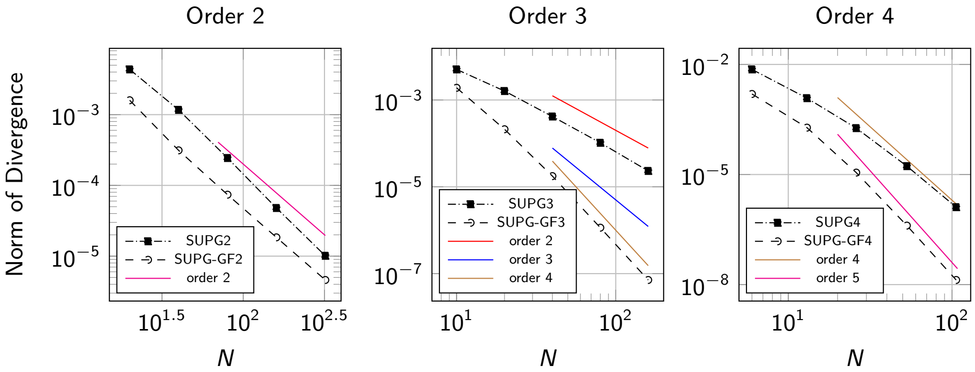


Figure: Smooth vortex: convergence of L^2 error of u with respect to the number of elements in x

Vortex perturbation

Pressure perturbation

- Gaussian centered in $\underline{x}_p = (0.4, 0.43)$
- scaling coefficient $r_0 = 0.1$
- radius $\rho(\underline{x}) = \sqrt{||\underline{x} - \underline{x}_p||}/r_0$

$$\delta_p(\underline{x}) = \varepsilon e^{-\frac{1}{2(1-\rho(\underline{x}))^2}+\frac{1}{2}},$$

- final time $T = 0.35$

Vortex perturbation

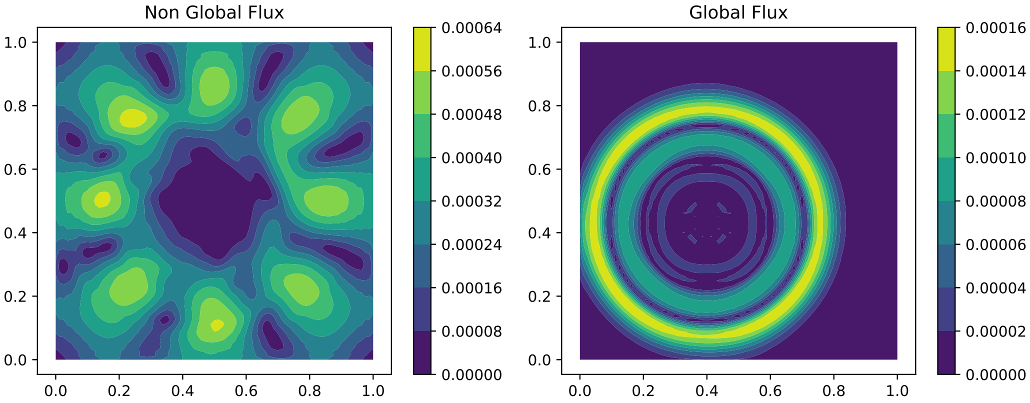


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^1$ with 80×80 cells and 6561 dofs.

Vortex perturbation

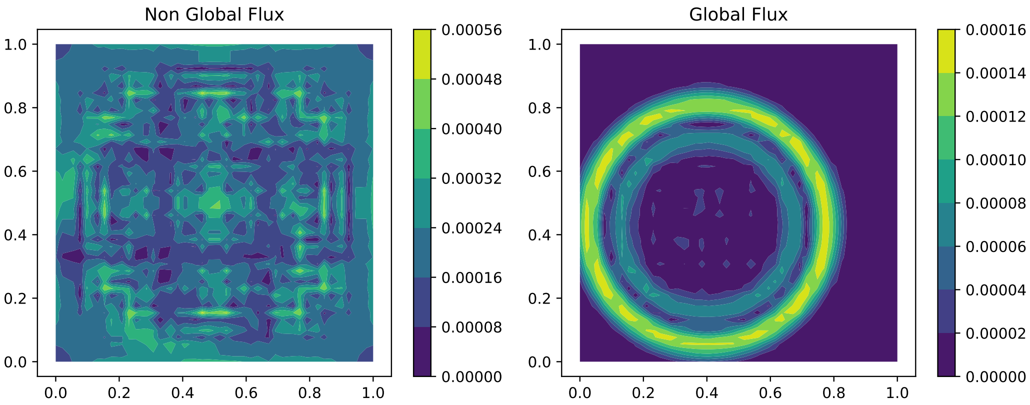


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 13×13 cells and 1600 dofs.

Vortex perturbation

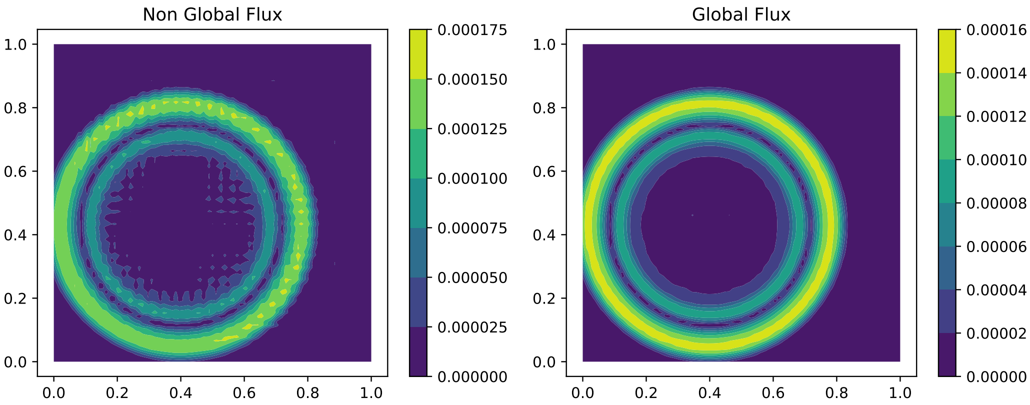


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 26 cells and 6241 dofs.

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Extensions and Perspectives

Extensions

- Source + divergence: funny systems in 2D
- Other methods (FV, CG, DG, FD)
- Other stabilizations (FEM + SUPG, FEM + OSS)

References

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Perspectives

- 2D nonlinear
- Non Cartesian meshes
- Curl-preserving
- Curl-free form MHD

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THANKS!!