Divergence-free Preserving Schemes: what's wrong in SUPG and how to fix it

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SUPG

SUPG

- Streamline **upwind** Petrov-Galerkin
- Stabilization for Continuous Galerkin Finite Element methods
- Advection dominated problems

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Remi's philosophy

"Everything is Residual Distribution"

FD=FV=FEM=DG

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Remi's philosophy

"Everything is Residual Distribution"

FD=FV=FEM=DG

with appropriate choice of:

- quadrature formula
- numerical flux
- interpretation of control volume
- definition of residuals
- etc.

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SUPG for a **system** of linear hyperbolic conservation laws

$$\partial_t q + J^x \partial_x q + J^y \partial_y q = 0, \qquad q : \Omega_h \times \mathbb{R}^+ \to \mathbb{R}^S.$$

Take $V_h^K := \{\varphi \in \mathcal{C}(\Omega_h) : \varphi|_E \in \mathbb{P}^K(E) \forall E \in \Omega_h\}$. SUPG is $\forall \varphi \in (V_{h,0}^K)^S$ find $q \in (V_h^K)^S$ such that

$$0 = \int (\varphi)\left(\partial_t q + J^x \partial_x q + J^y \partial_y q\right) \mathrm{d}x$$

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$$0 = \int (\varphi + \alpha \Delta x \partial_x \varphi \, \mathrm{sign}(J^x) + \alpha \Delta y \partial_y \varphi \, \mathrm{sign}(J^y)) \, (\partial_t q + J^x \partial_x q + J^y \partial_y q) \, \mathrm{d}x$$

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$$\begin{aligned}
0 &= \int \left(\varphi + \alpha \Delta x \partial_x \varphi \,\mathrm{sign}(J^x) + \alpha \Delta y \partial_y \varphi \,\mathrm{sign}(J^y)\right) \left(\partial_t q + J^x \partial_x q + J^y \partial_y q\right) \mathrm{d}x \\
&= \int \varphi(\partial_t q + J^x \partial_x q + J^y \partial_y q) \mathrm{d}x + \alpha \int (\Delta x \partial_x \varphi \,\mathrm{sign}(J^x) + \Delta y \partial_y \varphi \,\mathrm{sign}(J^y)) \partial_t q \,\mathrm{d}x \\
&+ \alpha \int \Delta x \partial_x \varphi \,\mathrm{sign}(J^x)(J^x \partial_x q + J^y \partial_y q) \mathrm{d}x + \alpha \int \Delta y \partial_y \varphi \,\mathrm{sign}(J^y)(J^x \partial_x q + J^y \partial_y q) \mathrm{d}x.
\end{aligned}$$

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Acoustics equations and involutions

Acoustics equation

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t v + \partial_y p = 0, \\ \partial_t p + \partial_x u + \partial_y v = 0, \end{cases}$$

$$\begin{cases} \partial_t \underline{v} + \nabla p = 0, \\ \partial_t p + \nabla \cdot \underline{v} = 0, \end{cases}$$

$$\partial_t q + J^x \partial_x q + J^y \partial_y q = 0$$

$$q = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \quad J^x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Involution

The system of linear acoustics possesses an involution:

$$\partial_t(\nabla \times \underline{v}) = \nabla \times \partial_t \underline{v} = -\nabla \times \nabla p = 0,$$

Equilibria

$$q : \partial_t q = 0, \qquad \begin{cases} \nabla \cdot \underline{v} = 0 \\ p \equiv C \in \mathbb{R} \end{cases}$$

Other equations sharing div-free equilibria

SW, Euler, Maxwell, low Mach

Typical problems

4.1 Low Mach number limit 81

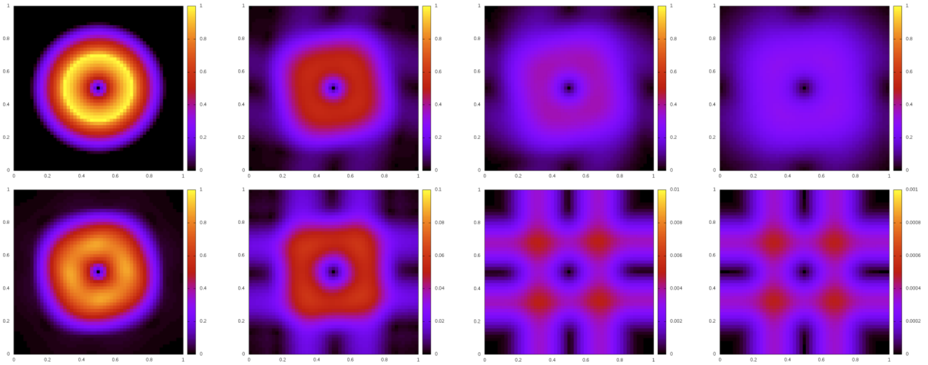

Figure 4.1: Simulation results for a vortex setup for $t = 0, 1, 2, 3$ (from left to right). Colour coded is $\sqrt{u^2 + v^2}$ *Top row*: Euler equations. *Bottom row*: Acoustic equations. [1][2]

[1] Barsukow, W. Low Mach number finite volume methods for the acoustic and Euler equations, Ph.D. thesis, 2018.
[2] Finite Volume Upwind numerical flux simulations.

Typical problems: SUPG

Let's try with SUPG.

Hope

$$\int \left(\varphi + \alpha \Delta x \partial_x \varphi \,\mathrm{sign}(J^x) + \alpha \Delta y \partial_y \varphi \,\mathrm{sign}(J^y)\right) \begin{pmatrix} \partial_t u + \partial_x p \\ \partial_t v + \partial_y p \\ \partial_t p + \partial_x u + \partial_y v \end{pmatrix} \mathrm{d}x = 0$$

Typical problems: SUPG

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$$\int (\varphi + \alpha\Delta x \partial_x \varphi \, \mathrm{sign}(J^x) + \alpha \Delta y \partial_y \varphi \, \mathrm{sign}(J^y)) \begin{pmatrix} \partial_t u + \partial_x p \\ \partial_t v + \partial_y p \\ \partial_t p + \partial_x u + \partial_y v \end{pmatrix} \mathrm{d}x = 0$$

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \Delta x \, \partial_x \varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha \Delta y \, \partial_y \varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t p + \partial_x u + \partial_y v) + \alpha \Delta x \, \partial_x \varphi(\partial_t u + \partial_x p) + \alpha \Delta y \, \partial_y \varphi(\partial_t v + \partial_y p) = 0$$

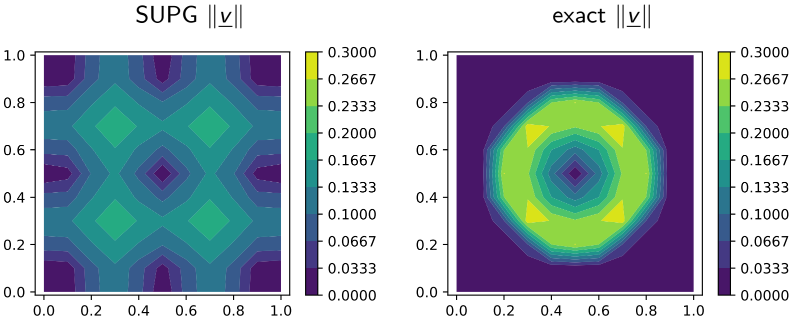
Typical problems: SUPG

Discretization

- Cartesian grid
- CG-FEM
- SUPG
- $\mathbb{Q}^1$
- $N_x = 10$
- $N_y = 10$

Test

- Vortex $\underline{v}$
- $p \equiv 1$
- Long time simulation $T = 100$



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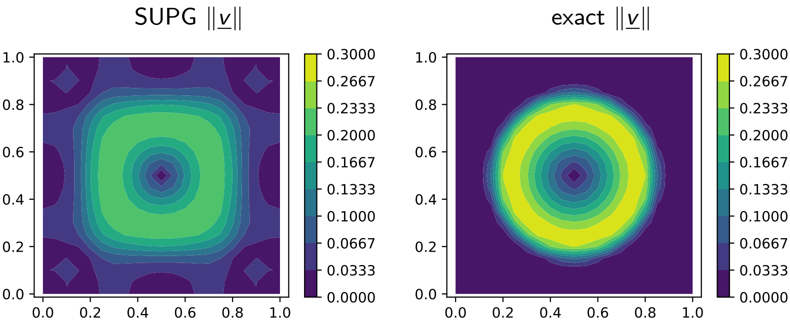

Typical problems: SUPG

Discretization

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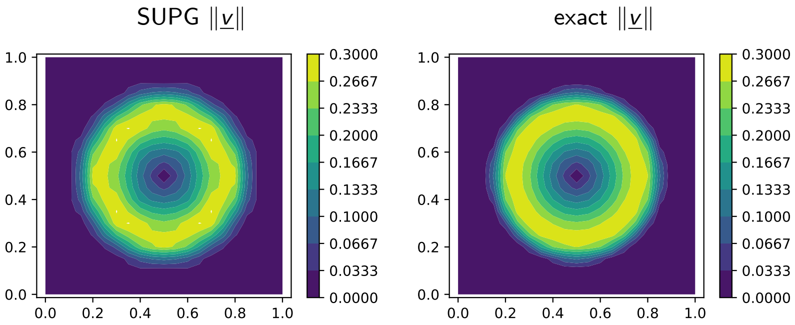

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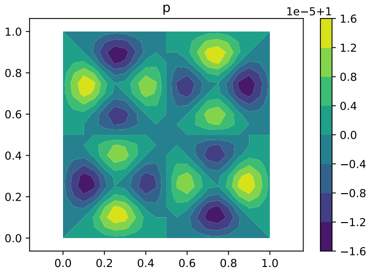
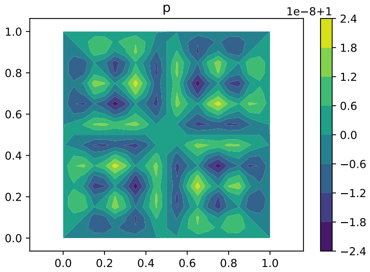
SUPG p, $\mathbb{Q}^1$, $N_x = N_y = 20$



exact p, $\mathbb{Q}^2$, $N_x = N_y = 10$



Why also SUPG?

Ideal

- At equilibrium $\partial_t q = 0$

SUPG formulation

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \Delta x \, \partial_x \varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

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- At equilibrium $\partial_t q = 0$
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- At equilibrium $\partial_t q = 0$
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- $\nabla \cdot \underline{v} = 0$
- $\int \varphi \nabla p = 0 \quad \forall \varphi \in V_{h,0}^K$
- $\int \nabla \varphi \cdot \nabla p = 0 \quad \forall \varphi \in V_{h,0}^K$

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Kernels

Example

$\int \varphi(\partial_x u + \partial_y v) = 0 \iff$

$$\mathcal{D}_x u + \mathcal{D}_y v = [\mathcal{D}_x \ \mathcal{D}_y] \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$\ker([\mathcal{D}_x \ \mathcal{D}_y]) \subset V_h^K \times V_h^K$

Kernel relation?

- $\ker \left[\int \varphi(\partial_x u + \partial_y v) \mathrm{dx} \quad \forall \varphi \in V_{h,0}^K \right]$
- $\ker \left[\int \partial_x \varphi(\partial_x u + \partial_y v) \mathrm{dx} \quad \forall \varphi \in V_{h,0}^K \right]$
- $\ker \left[\int \partial_Y \varphi(\partial_x u + \partial_y v) \mathrm{dx} \quad \forall \varphi \in V_{h,0}^K \right]$

Kernels

What are these kernels?

$$\ker\left(\int \varphi(\partial_x u + \partial_y v)\mathrm{d}x \quad \forall \varphi \in V_{h,0}^K\right) \qquad \ker\left(\int \partial_x\varphi(\partial_x u + \partial_y v)\mathrm{d}x \quad \forall \varphi \in V_{h,0}^K\right) \subset V_h^K \times V_h^K$$

$$\ker\left([\mathcal{D}_x \; \mathcal{D}_y]\right) \qquad \ker\left([\mathcal{D}_x^x \; \mathcal{D}_y^x]\right) \subset V_h^K \times V_h^K$$

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$$\ker\left([\mathcal{D}_x\ \mathcal{D}_y]\right) \qquad \ker\left([\mathcal{D}_x^x\ \mathcal{D}_y^x]\right) \subset V_h^K \times V_h^K$$

2D version

Not easy to study... But we can show that if

$$\int \varphi(\partial_x u + \partial_y v)\mathrm{d}x = 0 \quad \forall \varphi \in V_{h,0}^K \not\Longrightarrow \int \partial_x\varphi(\partial_x u + \partial_y v)\mathrm{d}x = 0 \quad \forall \varphi \in V_{h,0}^K$$

$$\ker\left[\int \varphi(\partial_x u + \partial_y v)\mathrm{d}x \quad \forall \varphi \in V_{h,0}^K\right] \not\subset \ker\left[\int \partial_x\varphi(\partial_x u + \partial_y v)\mathrm{d}x \quad \forall \varphi \in V_{h,0}^K\right]$$

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Kernel relations facts

$$\ker\left[\int \varphi(\partial_x u + \partial_y v)\mathrm{d}x \quad \forall \varphi \in V^K_{h,0}\right] \not\subset \ker\left[\int \partial_x\varphi(\partial_x u + \partial_y v)\mathrm{d}x \quad \forall \varphi \in V^K_{h,0}\right]$$

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- Is this good? Is this bad?

Kernel relations desiderata

- Divergence free solutions are in all kernels
  - If u, v are such that $\partial_x u + \partial_y v \equiv 0$: True

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$$\ker\left[\int \varphi(\partial_x u+\partial_y v)\mathrm{d}x \quad \forall \varphi\in V^K_{h,0}\right] \not\subset \ker\left[\int \partial_x\varphi(\partial_x u+\partial_y v)\mathrm{d}x \quad \forall \varphi\in V^K_{h,0}\right]$$

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- Other u, v are such that $\int \varphi(\partial_x u+\partial_y v)\mathrm{d}x=0 \quad \forall \varphi\in V^K_{h,0}$
- If u, v are not $\partial_x u+\partial_y v\not\equiv 0$ we would like them to be dissipated by the artificial viscosity terms

$$\left\{\ker\left[\int \varphi(\partial_x u+\partial_y v)\mathrm{d}x\,\forall \varphi\in V^K_{h,0}\right]\setminus[\partial_x u+\partial_y v\equiv 0]\right\}\cap \ker\left[\int \partial_y\varphi(\partial_x u+\partial_y v)\mathrm{d}x\,\forall \varphi\in V^K_{h,0}\right]=\emptyset$$

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How to solve the problem?

Recipe?

- 2D operators with more recognizable kernels
- Recast 2D operators to 1D operators to easily study their kernels
- Divergence operator that should be a Kronecher product of operators

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Global Flux

Reminder of what is Global Flux 1D (for balance laws)

$$\partial_t U + \partial_x F(U) = S(U)$$

$$G(U) := F(U) - \int^x S(U)$$

$$\partial_t U + \partial_x G(U) = 0$$

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Global Flux SUPG for acoustics

Define $\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds$ and $\boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$, with $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y \in V_h^K(\Omega_h)$, $\Phi := \boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y$.

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \Delta x\, \partial_x \varphi(\partial_t p + \partial_x \partial_y (\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y)) = 0$$

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$$\int \varphi(\partial_t u + \partial_x p) + \alpha \Delta x \, \partial_x \varphi(\partial_t p + \partial_x \partial_y \Phi) = 0$$

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Changes in equilibrium

$$\begin{aligned} &\nabla \cdot \underline{v} = 0 \\ \Longrightarrow &\partial_x \partial_y (\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0 \\ \Longleftrightarrow &\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y = f(x) + g(y) \end{aligned}$$

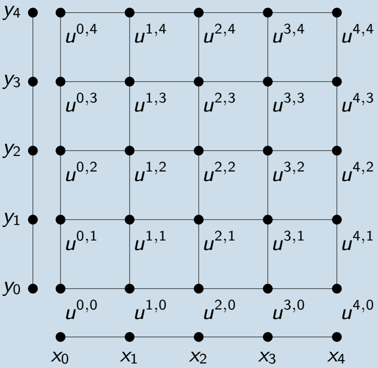
Discrete equilibrium

$$\begin{aligned} &\partial_x \partial_y \Phi(x_i, y_j) = 0 \\ \Longrightarrow &\int_{x_0}^{x_i} \int_{y_0}^{y_j} \partial_y \partial_x \Phi(x,y) dx dy = 0 \quad \forall i,j \\ \Longrightarrow &\int_{x_0}^{x_i} \partial_x \Phi(x, y_j) dx - \int_{x_0}^{x_i} \partial_x \Phi(x, y_0) dx = 0 \quad \forall i,j \\ \Longrightarrow &\Phi(x_i, y_j) - \Phi(x_0, y_j) - \Phi(x_i, y_0) + \Phi(x_0, y_0) = 0 \quad \forall i,j \\ \Longrightarrow &\Phi(x_i, y_j) = f_i + g_j \end{aligned}$$

Detailed definition of Global Flux SUPG

Definition of $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$

Cartesian grid, Lagrangian basis functions in Lobatto points (x_i, y_j) in each direction.
So, $\phi_i(x_k) = \delta_{ik}$ and $\psi_j(y_\ell) = \delta_{j\ell}$ and



$$u(x,y) = \sum_{i,j} \varphi_{ij}(x,y) u^{i,j} = \sum_{i,j} \phi_i(x)\psi_j(y) u^{ij}$$

$$u(x_i, y_j) = u^{i,j}$$

$$\boldsymbol{\sigma}_x(x,y) = \sum_{i,j} \phi_i(x)\psi_j(y)\boldsymbol{\sigma}_x^{i,j}$$

$$\boldsymbol{\sigma}_x(x_i, y_j) = \boldsymbol{\sigma}_x^{i,j}$$

$$\boldsymbol{\sigma}_x(x,y) = \int_{y_0}^{y} u(x,s)ds$$

$$\boldsymbol{\sigma}_x^{i,j} = \boldsymbol{\sigma}_x(x_i, y_j) = \int_{y_0}^{y_j} u(x_i, s)ds = \sum_{k,\ell} \phi_k(x_i) \int_{y_0}^{y_j} \psi_\ell(s)ds\, u^{k,\ell}$$

So, even if both $\boldsymbol{\sigma}_x, u \in V_h^K$, in quadrature points, we have that exactly $u(x_i, y_j) = \int_{y_0}^{y_j} \boldsymbol{\sigma}_x(x_i, y)dy$.

Myth buster

Global Flux is not global!

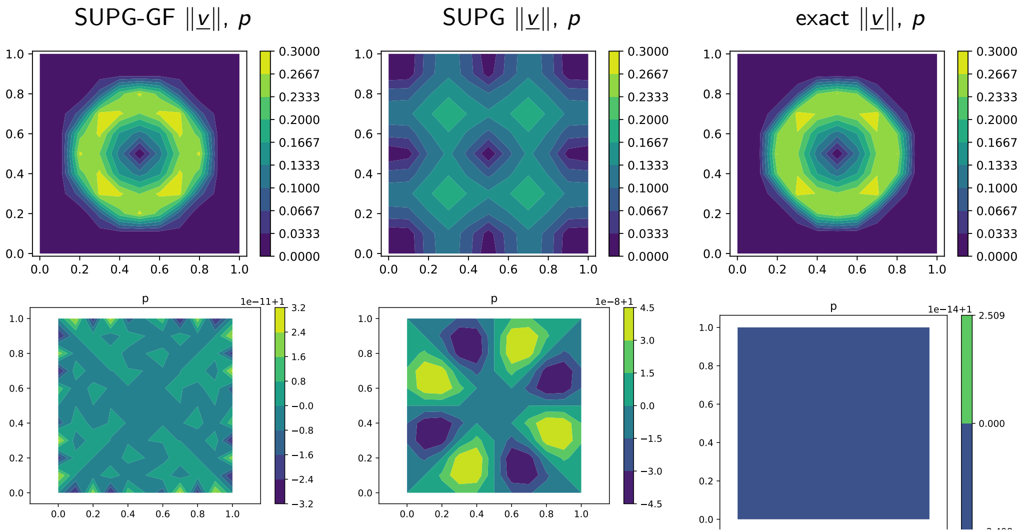
- In principle $\boldsymbol{\sigma}_x(x,y) = \int_{y_B}^{y} u(x,s)ds$ should be integrated from the beginning (bottom) of the domain y_B!
- In practice we always use $\partial_x \partial_y \boldsymbol{\sigma}_x(x,y)$ integrated in one cell!!!!
- So,

$$\boldsymbol{\sigma}_x(x,y) = \int_{y_B}^{y} u(x,s)ds = \underbrace{\int_{y_B}^{y_0} u(x,s)ds}_{\text{constant in one cell!}} + \int_{y_0}^{y} u(x,s)ds$$

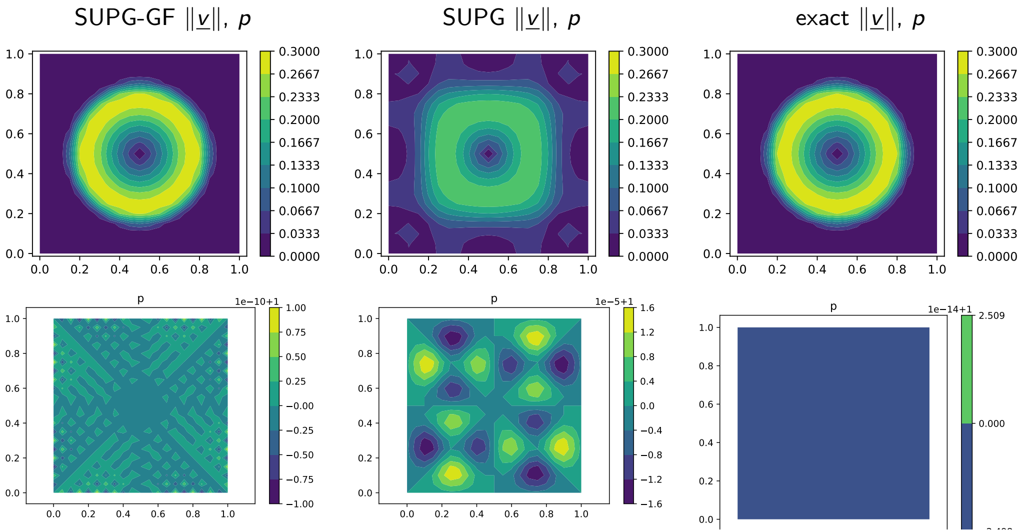
whatever constant we bring from outside the cell, is canceled out

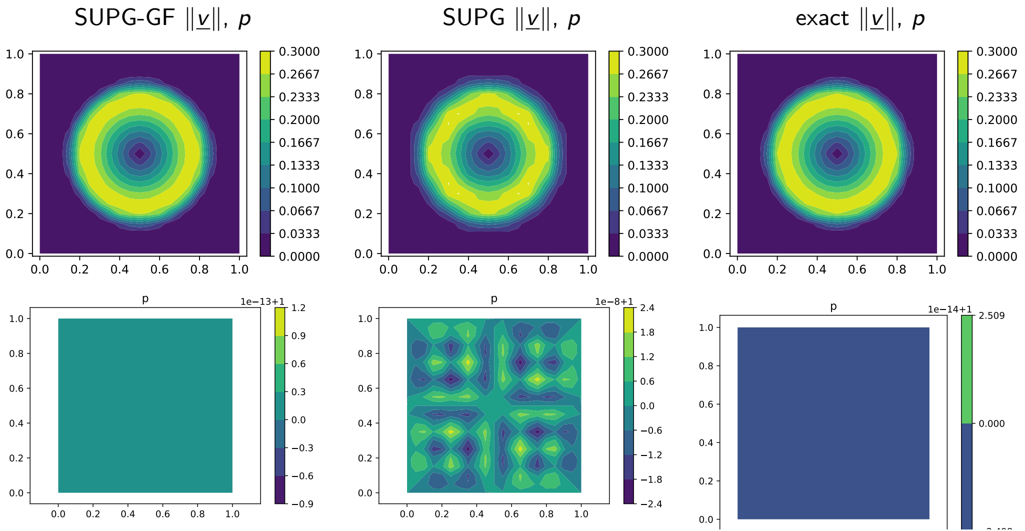
$$\partial_y \boldsymbol{\sigma}_x(x,y) = \partial_y \int_{y_B}^{y} u(x,s)ds = \partial_y \int_{y_B}^{y_0} u(x,s)ds + \partial_y \int_{y_0}^{y} u(x,s)ds = \partial_y \int_{y_0}^{y} u(x,s)ds$$

Simulation of vortex: $\mathbb{Q}^1$, $\quad N_x = N_y = 10$



Simulation of vortex: $\mathbb{Q}^1$, $N_x = N_y = 20$




SUPG-GF $\|\underline{v}\|$, p
SUPG $\|\underline{v}\|$, p
exact $\|\underline{v}\|$, p
0.3000
0.2667
0.2333
0.2000
0.1667
0.1333
0.1000
0.0667
0.0333
0.0000
p
1e−13+1
1e−8+1
1e−14+1
2.509
0.000

Simulation of vortex: errors

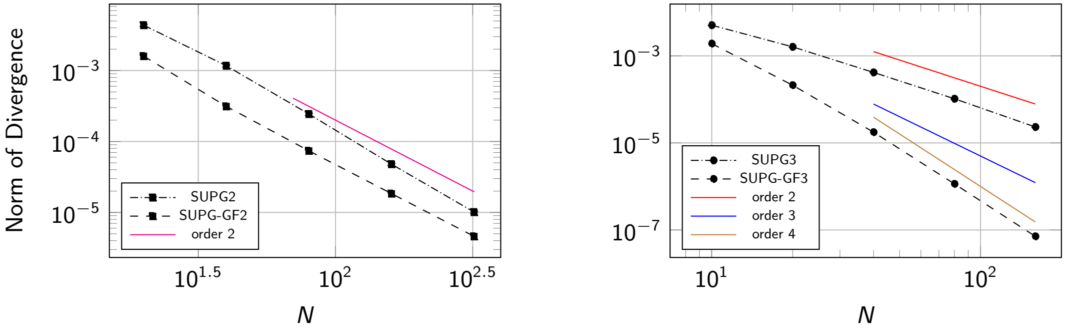


Figure: Smooth vortex: convergence of L^2 error of u with respect to the number of elements in x

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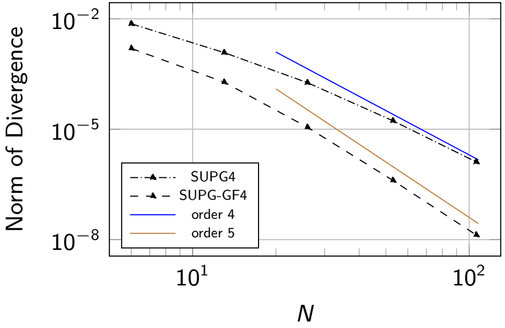

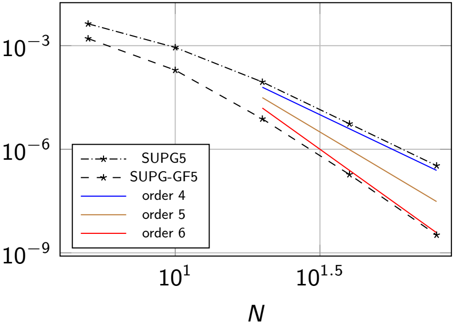

Figure: Smooth vortex: convergence of L^2 error of u with respect to the number of elements in x

Vortex simulation: divergence error

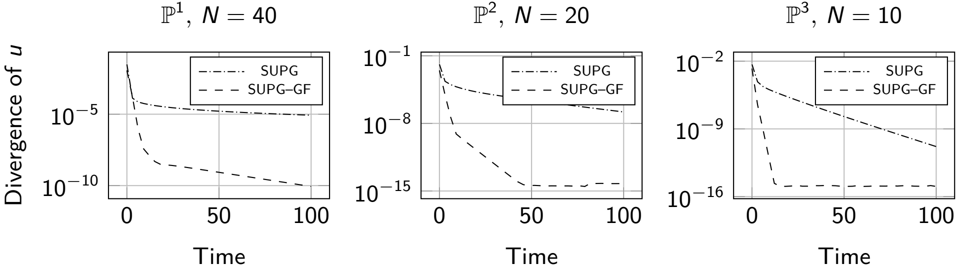

Figure: Norm of discrete divergence of $\underline{u}$ for SUPG ($\partial_x u + \partial_y v$) and SUPG–GF ($\partial_x \partial_y (\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y)$) simulations with respect to time for different orders

Table of contents

Why SUPG-GF works so better?

Clearly divergence-free preserving

- Which divergence? $\partial_x\partial_y(\boldsymbol{\sigma}_x+\boldsymbol{\sigma}_y)\approx \partial_x\partial_y\left(\int^y u(x,s)ds+\int^x v(s,y)ds\right)=\partial_x u+\partial_y v$

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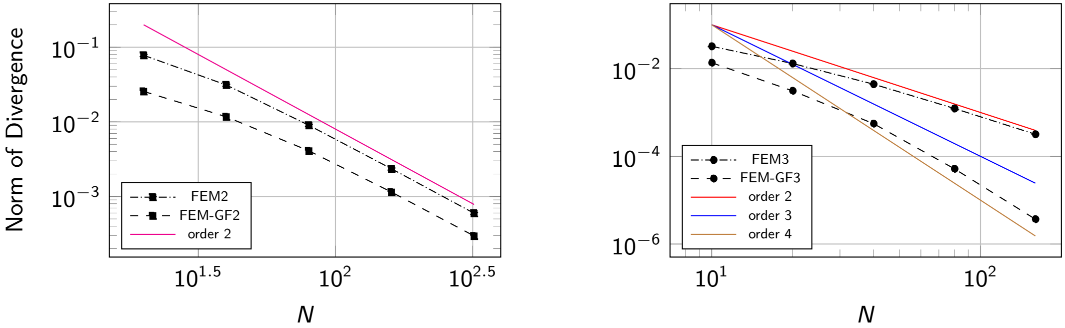

Figure: Smooth vortex: convergence of divergence operator on exact IC with respect to the number of elements in x

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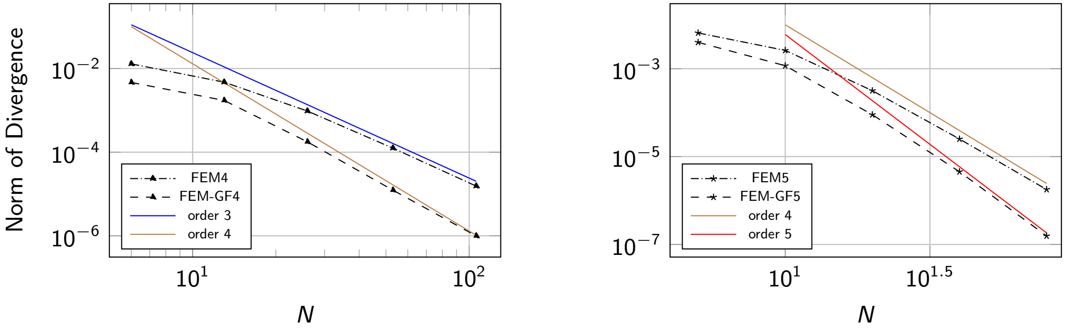

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- If we know that $\partial_x\partial_y(\boldsymbol{\sigma}_x+\boldsymbol{\sigma}_y) \equiv 0$ and $p \equiv c$ then equilibrium

Why SUPG-GF works so better?

New operators kernels

$$\boldsymbol{\Phi} = \boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y = \int^y u + \int^x v \in V_h^K$$

$$\int_{\Omega_h} \varphi(x,y)\partial_x\partial_y(\Phi)dxdy = 0, \; \forall \varphi \in V_{h,0}^K$$

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Matrix formulation

$$(D_x)_{ij} := \int \phi_i(x)\partial_x\phi_j(x)dx \qquad (D_x^x)_{ij} := \int \partial_x\phi_i(x)\partial_x\phi_j(x)dx$$

$$\boldsymbol{\Phi} = \boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y \qquad \boldsymbol{\Phi} \in \mathbb{R}^{(N_xK+1)\times(N_yK+1)}$$

$$(D_x \otimes D_y)\,\boldsymbol{\Phi} = 0 \qquad D_x, D_x^x \in \mathbb{R}^{(N_xK-1)\times(N_xK+1)}$$

$$(D_x^x \otimes D_y)\,\boldsymbol{\Phi} = 0 \qquad D_y, D_y^y \in \mathbb{R}^{(N_yK-1)\times(N_yK+1)}$$

$$(D_x \otimes D_y^y)\,\boldsymbol{\Phi} = 0$$

Kernels of Kronecher products

$$\ker(A_x \otimes B_y) = \ker(A_x) \otimes \mathbb{R}^{N_yK+1} + \mathbb{R}^{N_xK+1} \otimes \ker(B_y)$$

We can pass from the study of the 2D operators to the 1D operators!
Reminder: before it was not possible because we had a combination of operators $D_x \otimes M_y u + M_x \otimes D_y v = 0$.

Kernels from 2D to 1D

Kernels of Kronecher products

$$\ker(A_x \otimes B_y) = \ker(A_x) \otimes \mathbb{R}^{N_y K+1} + \mathbb{R}^{N_x K+1} \otimes \ker(B_y)$$

2D operators

Goal: study kernels of

- $D_x \otimes D_y$
- $D_x^x \otimes D_y$
- $D_x \otimes D_y^y$

in $\mathbb{R}^{N_x K+1} \times \mathbb{R}^{N_y K+1}$

1D operators

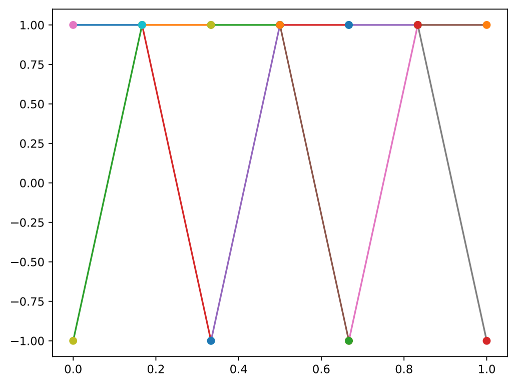
Study kernels of

- D_x
- D_x^x

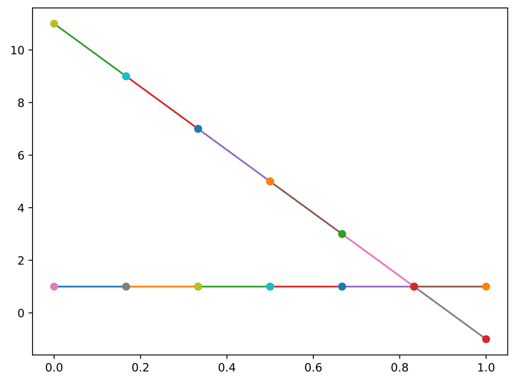
in $\mathbb{R}^{N_x K+1}$

One dimensional kernels of D_x and D_x^x

Kernel of D_x, 2 basis functions



Kernel of D_x^x, 2 basis functions



Operators

- Divergence $D_x \otimes D_y$
- Stabilization $D_x^x \otimes D_y$, $D_x \otimes D_y^y$

One dimensional kernels of D_x and D_x^x

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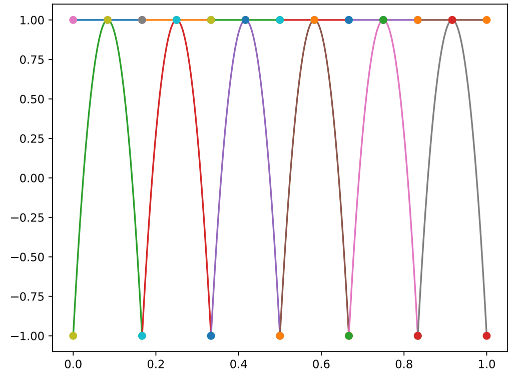

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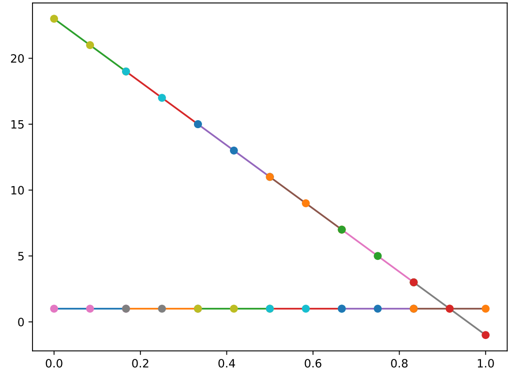

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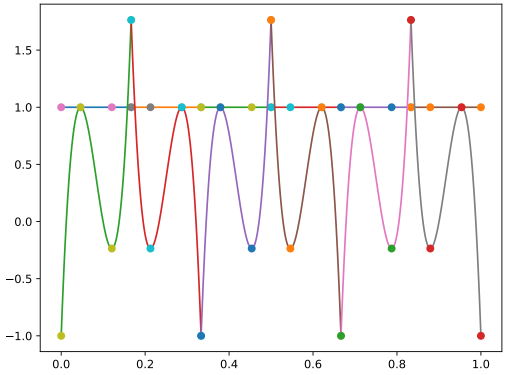

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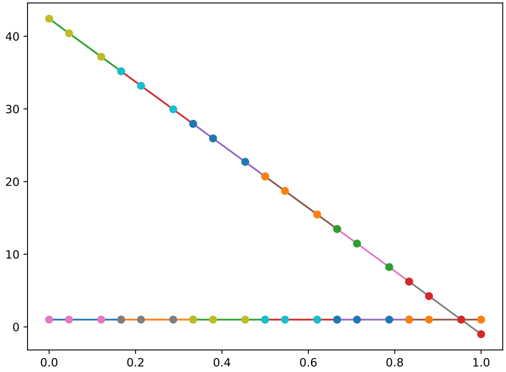

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Kernels of D_x, D_x^x and 2D operators

Theorem

- $\ker(D_x) = \text{span}\{1, w\}$ with w with first derivative discontinuous at each interface (chainsaw function)
- $\ker(D_x^x) = \text{span}\{1, x\}$

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- $(D_x \otimes D_y^y)(w \otimes g_2) = D_x w \otimes D_y^y g_2 = 0 \otimes * = 0$
- $(D_x^x \otimes D_y)(w \otimes g_2) = D_x^x w \otimes D_y g_2 \neq 0$ except
- $(D_x^x \otimes D_y)(w \otimes 1) = D_x^x w \otimes D_y 1 = 0$ and
- $(D_x^x \otimes D_y)(w \otimes w) = D_x^x w \otimes D_y w = 0$

Involution

Analytical involution

$$\nabla \times \partial_t \underline{v} = \nabla \times (\nabla p) = 0$$

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Discrete involution

- If linear method of line $\partial_t Q = F(Q) \Longrightarrow Q^{n+1} = M(F, Q^n)$
- Operator E such that $E \cdot (M(F,Q)) = M(E \cdot F, Q) = 0$ for all Q

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2D SUPG $\mathbb{Q}^1$ involution operator

$$E := \begin{pmatrix} D_x\Big((D_y)^2 M_x - \alpha^2 \Delta y D_y^y \big(\Delta y D_y^y M_x + \Delta x D_x^x M_y \big) \Big) \\ D_y\Big(-(D_x)^2 M_y + \alpha^2 \Delta x D_x^x \big(\Delta y D_y^y M_x + \Delta x D_x^x M_y \big) \Big) \\ \alpha \left(-\Delta x D_x^x (D_y)^2 M_x + (D_x)^2 \Delta y D_y^y M_y \right) \end{pmatrix}.$$

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- Operator E such that $E \cdot (M(F,Q)) = M(E \cdot F, Q) = 0$ for all Q

2D SUPG $\mathbb{Q}^1$ involution operator

$$E := \begin{pmatrix} D_x\left((D_y)^2 M_x - \alpha^2 \Delta y D_y^y \left(\Delta y D_y^y M_x + \Delta x D_x^x M_y\right)\right) \\ D_y\left(-(D_x)^2 M_y + \alpha^2 \Delta x D_x^x \left(\Delta y D_y^y M_x + \Delta x D_x^x M_y\right)\right) \\ \alpha\left(-\Delta x D_x^x (D_y)^2 M_x + (D_x)^2 \Delta y D_y^y M_y\right) \end{pmatrix}.$$

2D SUPG $\mathbb{Q}^p$ involution operator

AHAHAHAHAHAHAH

Properties summary

Discretization

- $u, v, p, \sigma_x, \sigma_y, \Phi \in V_h^K$

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- $p \equiv C$ and u, v:

$$\Phi_{ij} = \int_{y_0}^{y_j} u(x_i, y)dy + \int_{x_0}^{x_i} v(x, y_j)dx$$

- $\Phi_{ij} = f_i + g_j$
- $\Phi_{ij} - \Phi_{i0} - \Phi_{0j} + \Phi_{00} = 0$ for all i, j

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Dissipation of spurious modes

- Divergence operator $D_x \otimes D_y$ has spurious equilibria
- $D_x^x \otimes D_y$ or $D_x \otimes D_y^y$ dissipate all spurious equilibria
- Except a spurious mode $w \otimes w$ that is typically dissipated by boundary conditions or not present in IC

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Involution

- It is "possible" to compute the discrete involution, but not so nice

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FEM and quadrature

FEM details

- Lagrangian basis functions
- Gauss–Lobatto nodes for quadrature
- Gauss–Lobatto nodes for basis function
- Tensor product/Kronecher product to 2D structures

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SUPG-GF FEM discretization

$$
\begin{aligned}
&\Phi := \mathrm{Id}_x \otimes I_y\, u + I_x \otimes \mathrm{Id}_y\, v\\
0 =& M_x \otimes M_y \partial_t u + D_x \otimes M_y p + \alpha \Delta x \left(D^x \otimes M_y \partial_t p + D^x_x \otimes D_y I_y u + D^x_x I_x \otimes D_y v \right),\\
0 =& M_x \otimes M_y \partial_t v + M_x \otimes D_y p + \alpha \Delta y \left(M_x \otimes D^y \partial_t p + D_x \otimes D^y_y I_y u + D_x I_x \otimes D^y_y v \right),\\
0 =& M_x \otimes M_y \partial_t p + D_x \otimes D_y I_y u + D_x I_x \otimes D_y v +\\
&\alpha \left(\Delta x\, D^x \otimes M_y \partial_t u + \Delta y\, M_x \otimes D^y \partial_t v + (\Delta x\, D^x_x \otimes M_y + \Delta y\, M_x \otimes D^y_y) p \right).
\end{aligned}
$$

FEM and quadrature

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$$\begin{aligned}
\Phi &:= \mathrm{Id}_x \otimes I_y\, u + I_x \otimes \mathrm{Id}_y\, v\\
0 &= M_x \otimes M_y \partial_t u + D_x \otimes M_y p + \alpha \Delta x \left(D^x \otimes M_y \partial_t p + D_x^x \otimes D_y \Phi(u,v) \right),\\
0 &= M_x \otimes M_y \partial_t v + M_x \otimes D_y p + \alpha \Delta y \left(M_x \otimes D^y \partial_t p + D_x \otimes D_y^y \Phi(u,v) \right),\\
0 &= M_x \otimes M_y \partial_t p + D_x \otimes D_y \Phi(u,v) +\\
&\alpha \left(\Delta x\, D^x \otimes M_y \partial_t u + \Delta y\, M_x \otimes D^y \partial_t v + (\Delta x\, D_x^x \otimes M_y + \Delta y\, M_x \otimes D_y^y) p \right).
\end{aligned}$$

Table of contents

Smooth nonstationary test: oblique flow

Safety check!

Convergence of method on nonstationary problem with exact solution

$$\begin{cases} u(x,y,t) = -\frac{1}{2c}\left(\cos(\alpha\xi(x,y)+ct) - \cos(\alpha\xi(x,y)-ct)\right)\cos(\theta), \\ v(x,y,t) = -\frac{1}{2c}\left(\cos(\alpha\xi(x,y)+ct) - \cos(\alpha\xi(x,y)-ct)\right)\sin(\theta), \\ p(x,y,t) = \frac{1}{2}\left(\cos(\alpha\xi(x,y)+ct) + \cos(\alpha\xi(x,y)-ct)\right), \end{cases}$$

Smooth nonstationary test: oblique flow

Safety check!

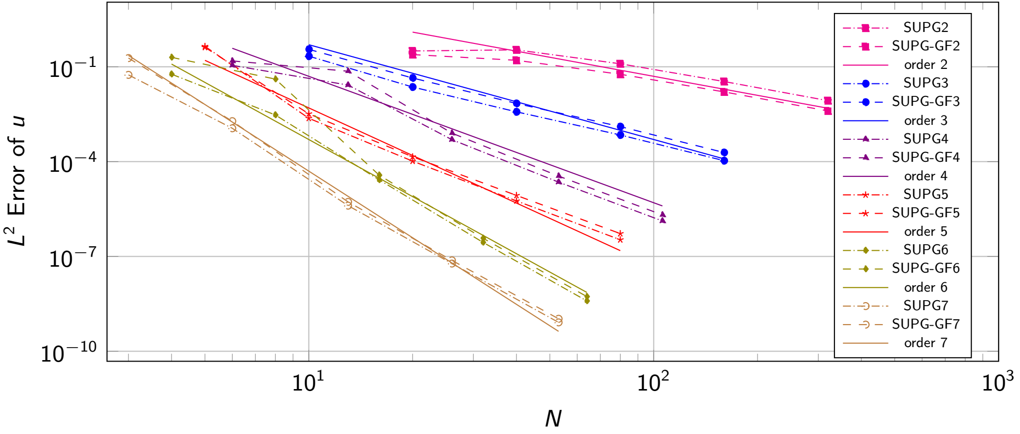


Figure: Oblique flow: convergence of L^2 error of u with respect to the number of elements in x

2D Riemann Problem

- Center $\underline{x}_0 = (0.5, 0.5)$
- Domain $\Omega = [0,1]^2$
- ICs

$$u(\underline{x}) = \begin{cases} 1, & \text{if } x > 0.5 \text{ and } y > 0.5, \\ 0, & \text{else,} \end{cases} \qquad v(\underline{x}) = 0, \qquad p(\underline{x}) = 0.$$

- The perpendicular component v has a logarithmic singularity in the center of the RP for all $t > 0$:

$$v(x,y,t) = \frac{1}{2\pi}\mathcal{L}\left(\frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{ct}\right),$$

$$\mathcal{L}(s) := \log\left(\frac{1+\sqrt{1-s^2}}{s}\right) = -\log\left(\frac{s}{2}\right) - \frac{s^2}{4} + \mathcal{O}(s^4).$$

2D Riemann Problem

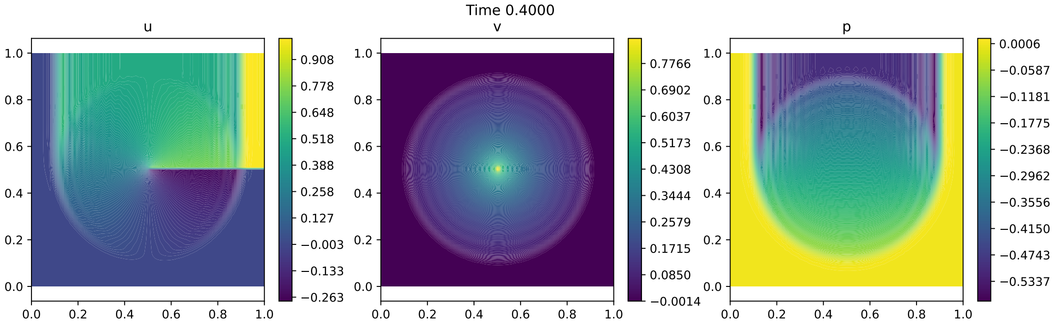


Figure: Riemann Problem. Simulation at time $T = 0.4$ with $\mathbb{P}^2$ elements and 50×50 cells with SUPG–GF scheme

2D Riemann Problem

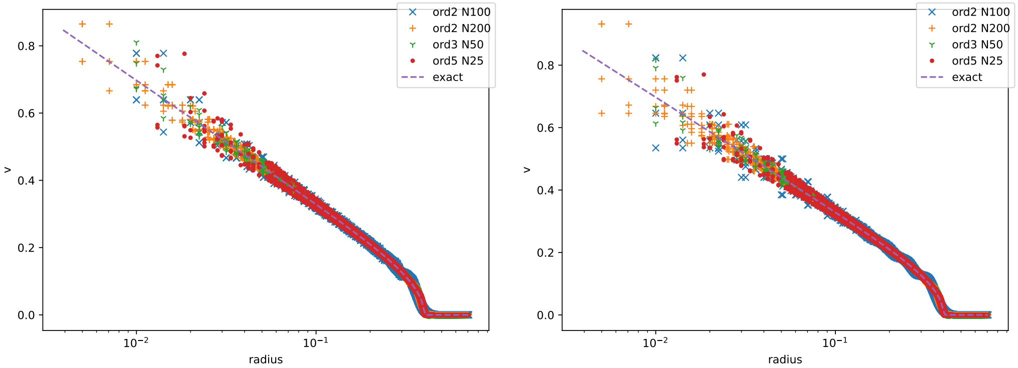


Figure: Riemann Problem. Distribution of the solution v for different elements and meshes. Left SUPG scheme, right SUPG–GF scheme

Vortex $\mathbb{Q}^2$ $N_x = N_y = 10$

$$\begin{cases} u(x,y) = f(\rho(x,y) \cdot (y - y_0) \\ v(x,y) = -f(\rho(x,y)) \cdot (x - x_0) \\ p(x,y) = 1 \end{cases}$$

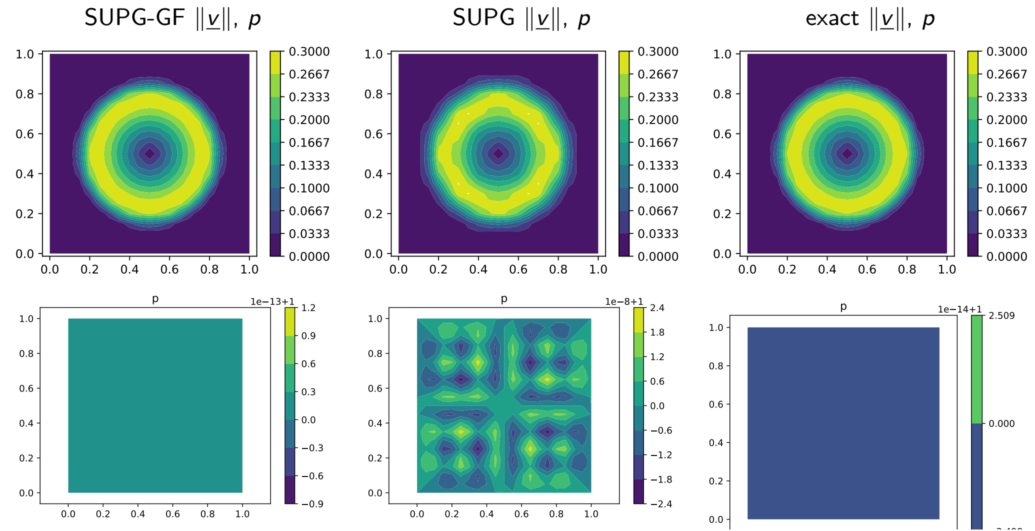
with $\rho(x,y) = \frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{r_0}$ with $r_0 = 0.45$ the radius of the support.

$$f(\rho) = 2\gamma e^{-\frac{1}{2(1-\rho)^2}} \sqrt{\frac{g}{r_0(1-\rho)^3}}$$

with $g = 9.81$, $\gamma = 0.2$ if $\rho < 1$, else 0.

$$T = 100$$

Vortex $\mathbb{Q}^2$ $N_x = N_y = 10$



Vortex perturbation

Pressure perturbation

- Gaussian centered in $\underline{x}_p = (0.4, 0.43)$
- scaling coefficient $r_0 = 0.1$
- radius $\rho(\underline{x}) = \sqrt{||\underline{x} - \underline{x}_p||}/r_0$

$$\delta_p(\underline{x}) = \varepsilon e^{-\frac{1}{2(1-\rho(\underline{x}))^2}+\frac{1}{2}},$$

- final time $T = 0.35$

Vortex perturbation

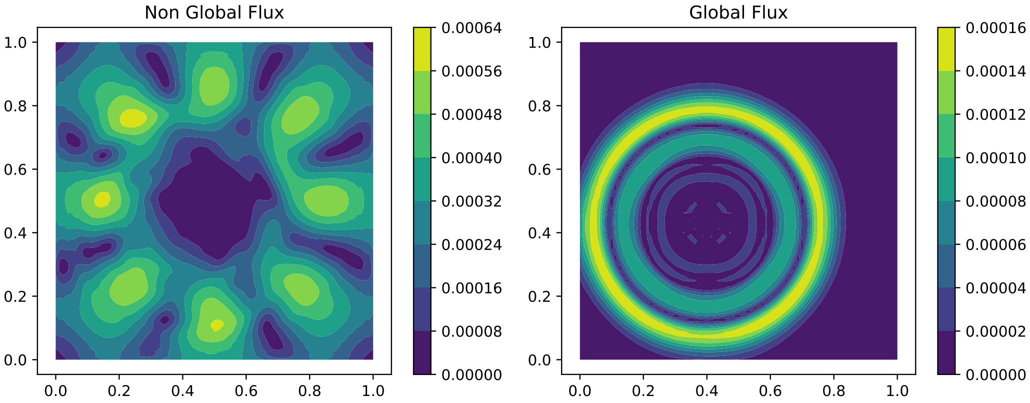


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^1$ with 80×80 cells and 6561 dofs.

Vortex perturbation

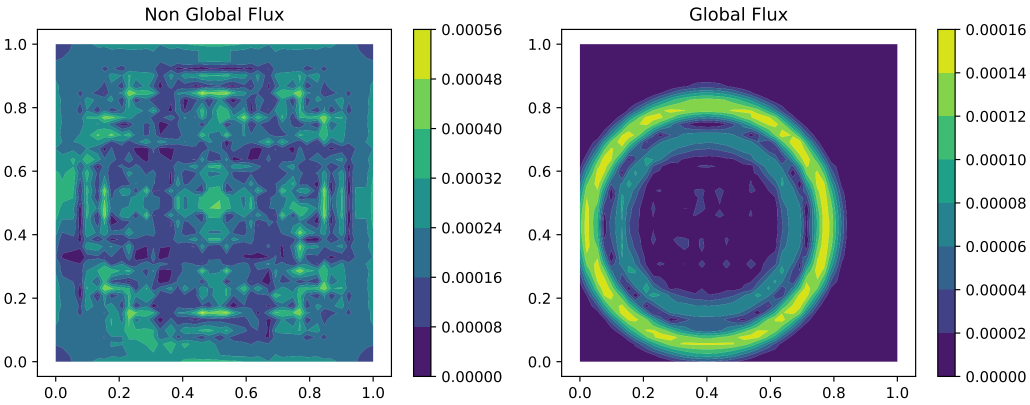


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 13×13 cells and 1600 dofs.

Vortex perturbation

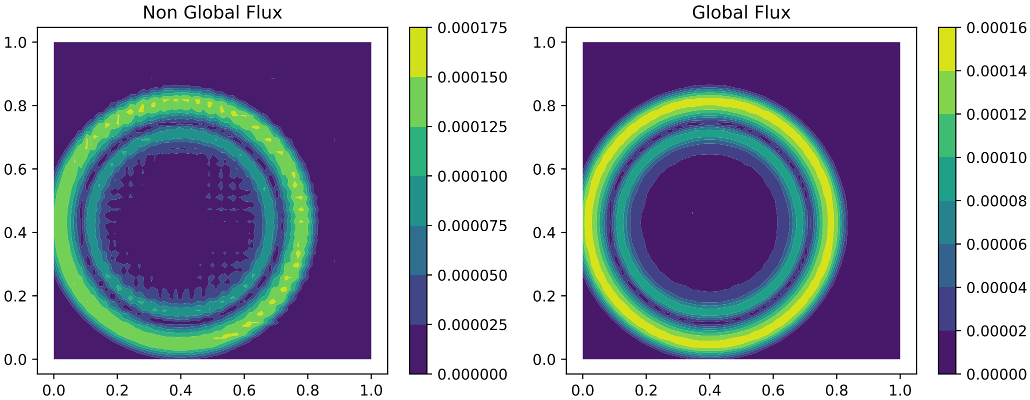


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 26 cells and 6241 dofs.

Vortex with Coriolis

Acoustic with Coriolis

$$\partial_t \begin{pmatrix} u \\ v \\ p \end{pmatrix} + \partial_x \begin{pmatrix} p \\ 0 \\ u \end{pmatrix} + \partial_y \begin{pmatrix} 0 \\ p \\ v \end{pmatrix} + c_f \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix} = 0.$$

GF for Acoustic with Coriolis

$$\partial_t \begin{pmatrix} u \\ v \\ p \end{pmatrix} + \begin{pmatrix} \partial_x(p + c_f \boldsymbol{\sigma}_y) \\ \partial_y(p - c_f \boldsymbol{\sigma}_x) \\ \partial_x \partial_y(\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) \end{pmatrix} = 0$$

FEM change

$$T_u^{2,m}(\underline{\underline{q}}) {+}{=} {-}c_f M_x \otimes M_y v^m$$

$$T_v^{2,m}(\underline{\underline{q}}) {+}{=} c_f M_x \otimes M_y u^m$$

$$T_p^{2,m}(\underline{\underline{q}}) {+}{=} c_f \alpha \left(M_x \otimes D^y u - D^x \otimes M_y v\right)$$

GF-FEM change

$$T_u^{2,m}(\underline{\underline{q}}) {+}{=} {-}c_f D_x I_x \otimes M_x v^m$$

$$T_v^{2,m}(\underline{\underline{q}}) {+}{=} c_f M_x \otimes D_y I_y u^m$$

$$T_p^{2,m}(\underline{\underline{q}}) {+}{=} c_f \alpha \left(M_x \otimes D_y^y I_y u - D_x^x I_x \otimes M_y v\right)$$

Vortex with Coriolis

Test

- $$\begin{cases} u(x,y) = -f(\rho(x,y)) \cdot (y-y_0), \\ v(x,y) = f(\rho(x,y)) \cdot (x-x_0), \\ p(x,y) = 1 - c_f \cdot g(\rho(x,y)), \end{cases}$$
- $\rho(x,y) = \sqrt{x^2+y^2}$
- $f(\rho) := 20e^{-100\rho^2}$
- $g(\rho) := \frac{1}{10}e^{-100\rho^2}$
- Domain $\Omega = [0,1]^2$

Vortex with Coriolis

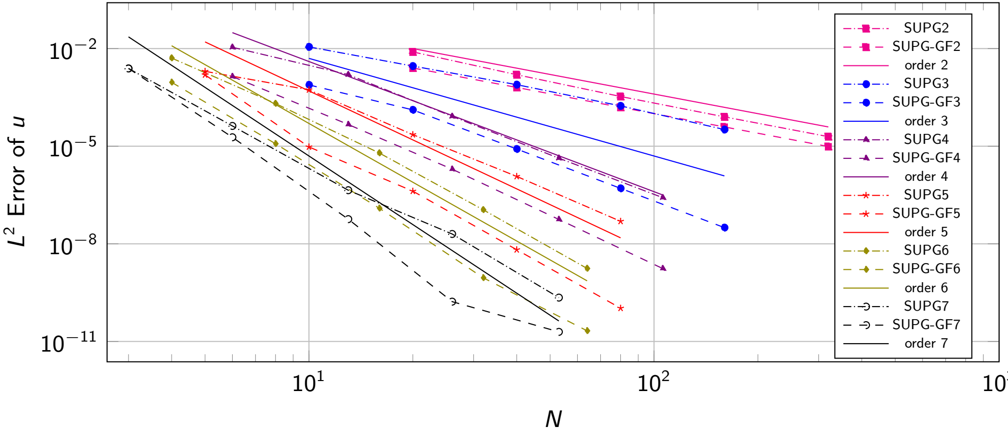
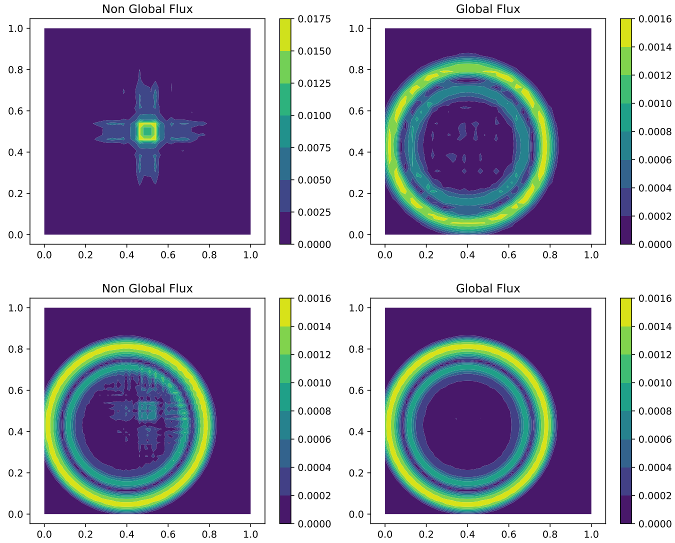


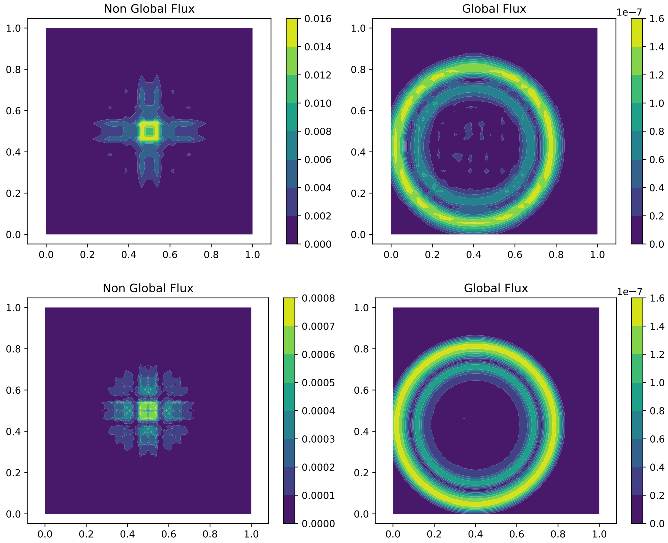
Figure: Coriolis vortex: convergence of L^2 error of u with respect to the number of elements in x

Vortex with Coriolis



Perturbation($\varepsilon = 10^{-2}$) test.
Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the analytical equilibrium.
Top $\mathbb{P}^3$ with 13 cells, bottom $\mathbb{P}^3$ with 26 cells.

Vortex with Coriolis



Perturbation($\varepsilon = 10^{-6}$) test.
Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the analytical equilibrium.
Top $\mathbb{P}^3$ with 13 cells, bottom $\mathbb{P}^3$ with 26 cells.

Source term

Consider the source equations

$$\begin{cases} \partial_t \underline{u} + \nabla p = 0, \\ \partial_t p + \nabla \cdot \underline{u} = s, \end{cases}$$

where an equilibrium solution can be found as

$$\begin{cases} p(x,y) \equiv p_0 \in \mathbb{R}, \\ \underline{u}(x,y) = \nabla^{\perp}\phi_1(x,y) + \nabla \phi_2(x,y), \\ s(x,y) = \Delta \phi_2(x,y), \end{cases}$$

for ϕ_1, ϕ_2 smooth enough. The first term of the velocity, i.e., $\nabla^{\perp}\phi_1(x,y)$ is analogous to the vortexes and it is divergence-free, while the second term and the source terms balance each other. We will consider the smooth steady vortex (31) for the first part of $\underline{u}$, while we will use $\phi_2(x,y) := \frac{1}{100} e^{-100||\underline{x}-\underline{x_0}||_2^2}$, with $\underline{x_0} = (0.65, 0.39)^T$.

Source term

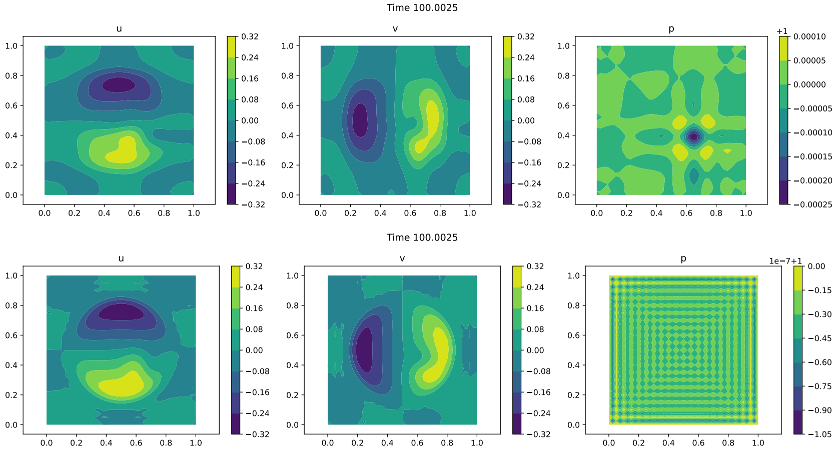


Figure: Simulation of vortex with source term at time $T = 100$ with $\mathbb{P}^1$ elements and 40×40 cells. SUPG scheme (top) and SUPG–GF scheme (bottom)

Source term

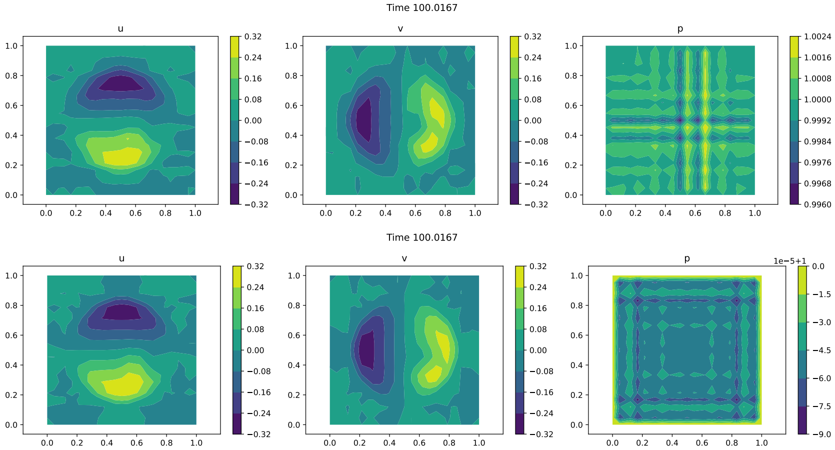
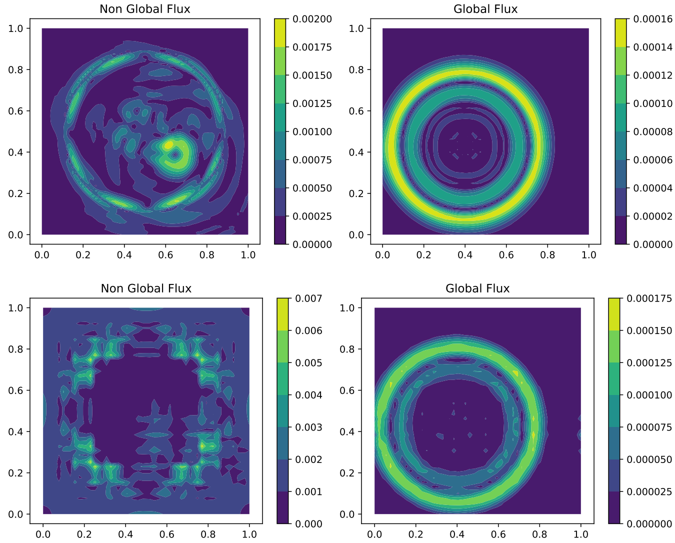


Figure: Simulation of vortex with source term at time $T = 100$ with $\mathbb{P}^3$ elements and 6×6 cells. SUPG scheme (top) and SUPG–GF scheme (bottom)

Source term



Vortex with Source

- Perturbation($\varepsilon = 10^{-3}$) test with source term.
- Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$
- Top $\mathbb{P}^1$ with 80×80 cells
- Bottom $\mathbb{P}^3$ with 13 cells

Stommel Gyre

Stommel Gyre

$$\partial_t p = -\operatorname{div}\mathbf{u}$$

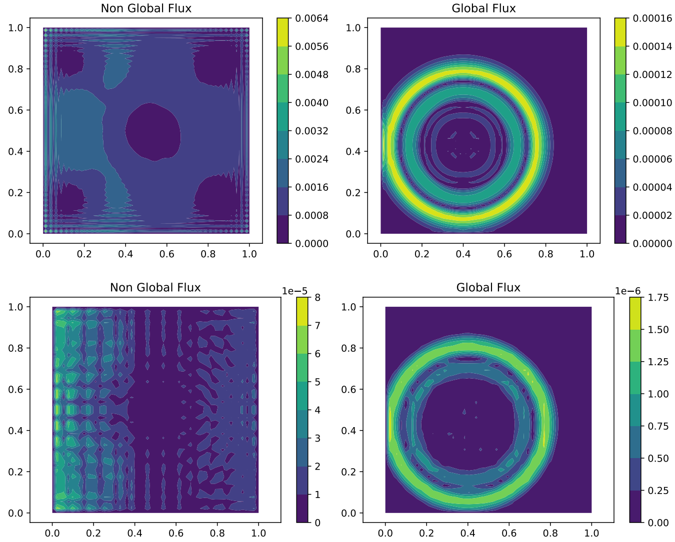
$$\partial_t \mathbf{u} = -\operatorname{grad} p + \phi \mathbf{u}^\perp - R\mathbf{u} + \boldsymbol{\tau}$$

Parameters

- $-R\mathbf{u}$ friction
- $\boldsymbol{\tau}$ wind forcing
- linearized shallow water equations, with a reference depth $h_0 = 1$, and with the gravity acceleration $g = 1$
- R is constant
- $\boldsymbol{\tau} = (-F\cos(\pi y/b), 0)$
- F constant
- well known steady solution due to Henry Stommel[a]

[a] H. Stommel, The westwards intensification of wind-driven ocean currents, Trans.Amer.Geophys.Union 29(2), 1948

Stommel Gyre



Perturbation test for SG

- Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$
- Top $\mathbb{P}^1$ with 80×80 cells $\varepsilon = 10^{-3}$
- Bottom $\mathbb{P}^3$ with 13 cells $\varepsilon = 10^{-5}$

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Orthogonal subscale stabilization

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \partial_x \varphi(\nabla \cdot \underline{v} - w^{\nabla\cdot\underline{v}}) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha \partial_y \varphi(\nabla \cdot \underline{v} - w^{\nabla\cdot\underline{v}}) = 0$$

$$\int \varphi(\partial_t p + \nabla \cdot \underline{v}) + \alpha \partial_x \varphi(\partial_x p - w_x^p) + \alpha \partial_y \varphi(\partial_y p - w_y^p) = 0$$

Projection definitions

$$\int \varphi w^p := \int \varphi \nabla p$$

$$\int \varphi w^{\nabla\cdot\underline{v}} := \int \varphi \nabla \cdot \underline{v}$$

Other stabilization: OSS

Orthogonal subscale stabilization

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \partial_x \varphi(\nabla \cdot \underline{v} - w^{\nabla\cdot \underline{v}}) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha \partial_y \varphi(\nabla \cdot \underline{v} - w^{\nabla\cdot \underline{v}}) = 0$$

$$\int \varphi(\partial_t p + \nabla \cdot \underline{v}) + \alpha \partial_x \varphi(\partial_x p - w^p_x) + \alpha \partial_y \varphi(\partial_y p - w^p_y) = 0$$

Projection definitions

$$\int \varphi w^p := \int \varphi \nabla p$$

$$\int \varphi w^{\nabla\cdot \underline{v}} := \int \varphi \nabla \cdot \underline{v}$$

GF-Orthogonal subscale stabilization

$$\int \varphi(\partial_t u + \partial_x p) + \alpha \partial_x \varphi(\partial_x \partial_y \Phi - w^{\partial_x \partial_y \Phi}) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha \partial_y \varphi(\partial_x \partial_y \Phi - w^{\partial_x \partial_y \Phi}) = 0$$

$$\int \varphi(\partial_t p + \partial_x \partial_y \Phi) + \alpha \partial_x \varphi(\partial_x p - w^p_x) + \alpha \partial_y \varphi(\partial_y p - w^p_y) = 0$$

$$\Phi := \int^y u + \int^x v$$

$$\int \varphi w^p := \int \varphi \nabla p$$

$$\int \varphi w^{\partial_x \partial_y \Phi} := \int \varphi \partial_x \partial_y \Phi$$

Non Cartesian meshes

Triangular meshes

- Haven't tried yet
- In principle, we can still define $\Phi := \int^y u + \int^x v$ in each element
- Question: will it be that effective?
- Kernels? Maybe difficult to write, still working

THANKS!!

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FEM and quadrature

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Matrices

- $(M_x)_{i,j} = \int \phi_i(x)\phi_j(x)dx$
- $(D_x)_{i,j} = \int \phi_i(x)\partial_x\phi_j(x)dx$
- $(D_x^x)_{i,j} = \int \partial_x\phi_i(x)\partial_x\phi_j(x)dx$
- $(I_x)_{i,j} = \int_{x_0}^{x_i} \phi_j(x)dx$

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- $(I_x)_{i,j} = \int_{x_0}^{x_i} \phi_j(x)dx$

SUPG-GF FEM discretization

$$
\begin{aligned}
\Phi &:= \mathrm{Id}_x \otimes I_y\, u + I_x \otimes \mathrm{Id}_y\, v\\
0 =& M_x \otimes M_y \partial_t u + D_x \otimes M_y p + \alpha h \left(D^x \otimes M_y \partial_t p + D_x^x \otimes D_y I_y u + D_x^x I_x \otimes D_y v\right),\\
0 =& M_x \otimes M_y \partial_t v + M_x \otimes D_y p + \alpha h \left(M_x \otimes D^y \partial_t p + D_x \otimes D_y^y I_y u + D_x I_x \otimes D_y^y v\right),\\
0 =& M_x \otimes M_y \partial_t p + D_x \otimes D_y I_y u + D_x I_x \otimes D_y v +\\
& \alpha h \left(D^x \otimes M_y \partial_t u + M_x \otimes D^y \partial_t v + (D_x^x \otimes M + M \otimes D_y^y) p\right).
\end{aligned}
$$

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SUPG-GF FEM discretization

$$
\begin{aligned}
\Phi &:= \mathrm{Id}_x \otimes I_y\, u + I_x \otimes \mathrm{Id}_y\, v \\
0 =& M_x \otimes M_y \partial_t u + D_x \otimes M_y p + \alpha h \left(D^x \otimes M_y \partial_t p + D_x^x \otimes D_y \Phi\right), \\
0 =& M_x \otimes M_y \partial_t v + M_x \otimes D_y p + \alpha h \left(M_x \otimes D^y \partial_t p + D_x \otimes D_y^y \Phi\right), \\
0 =& M_x \otimes M_y \partial_t p + D_x \otimes D_y \Phi + \\
& \alpha h \left(D^x \otimes M_y \partial_t u + M_x \otimes D^y \partial_t v + (D_x^x \otimes M + M \otimes D_y^y) p\right).
\end{aligned}
$$

Deferred Correction Iterative procedure

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

- $T^1(\underline{\boldsymbol{c}}) = 0$, first order accuracy, easily invertible.
- $T^2(\underline{\boldsymbol{c}}) = 0$, high order Q.

$$\boldsymbol{c}^{0,(p)} := \boldsymbol{c}(t_n), \quad p = 0, \dots, P,$$
$$\boldsymbol{c}^{m,(0)} := \boldsymbol{c}(t_n), \quad m = 1, \dots, M$$
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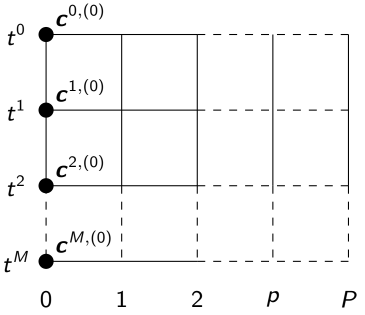

DeC Theorem

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- $T^1 - T^2$ Lipschitz with constant $\mathcal{O}(\Delta t)$

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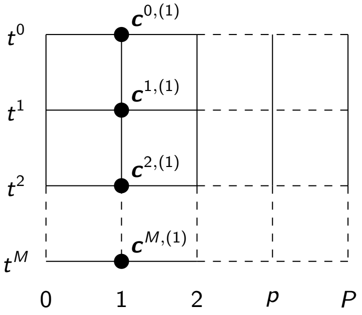

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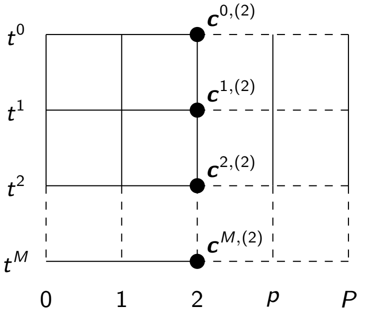

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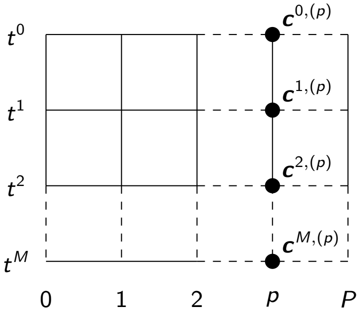

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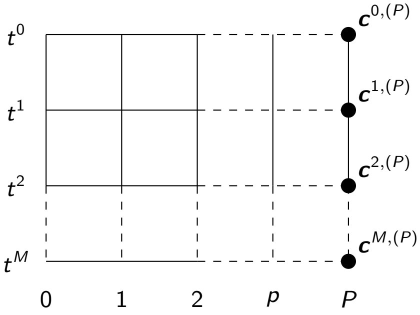

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DeC for SUPG

$$
\begin{aligned}
T_u^{2,m}(\underline{\underline{q}}) =& M_x \otimes M_y \frac{u^m - u^0}{\Delta t} + D_x \otimes M_y \sum_r \theta_r^m p^r + \\
& \alpha h (D^x \otimes M_y \frac{p^m - p^0}{\Delta t} + D_x^x \otimes D_y I_y \sum_r \theta_r^m u^r + D_x^x I_x \otimes D_y \sum_r \theta_r^m v^r), \\
T_v^{2,m}(\underline{\underline{q}}) =& M_x \otimes M_y \frac{v^m - v^0}{\Delta t} + M_x \otimes D_y \sum_r \theta_r^m p^r + \\
& \alpha h (M_x \otimes D^y \frac{p^m - p^0}{\Delta t} + D_x \otimes D_y^y I_y \sum_r \theta_r^m u^r + D_x I_x \otimes D_y^y \sum_r \theta_r^m v^r), \\
T_p^{2,m}(\underline{\underline{q}}) =& M_x \otimes M_y \frac{p^m - p^0}{\Delta t} + D_x \otimes D_y I_y \sum_r \theta_r^m u^r + D_x I_x \otimes D_y \sum_r \theta_r^m v^r + \\
& \alpha h (D^x \otimes M_y \frac{u^m - u^0}{\Delta t} + M_x \otimes D^y \frac{v^m - v^0}{\Delta t} + (D_x^x \otimes M_y + M_x \otimes D_y^y) \sum_r \theta_r^m p^r).
\end{aligned}
$$

DeC for SUPG

$$T_u^{1,m}(\underline{\underline{q}}) = M_x \otimes M_y \frac{u^m - u^0}{\Delta t} + \beta^m D_x \otimes M_y p^0,$$

$$T_v^{1,m}(\underline{\underline{q}}) = M_x \otimes M_y \frac{v^m - v^0}{\Delta t} + \beta^m M_x \otimes D_y p^0,$$

$$T_p^{1,m}(\underline{\underline{q}}) = M_x \otimes M_y \frac{p^m - p^0}{\Delta t} + \beta^m (D_x \otimes D_y I_y u^0 + D_x I_x \otimes D_y v^0).$$

$$T_u^{2,m}(\underline{\underline{q}}) = M_x \otimes M_y \frac{u^m - u^0}{\Delta t} + D_x \otimes M_y \sum_r \theta_r^m p^r +$$
$$\alpha h (D^x \otimes M_y \frac{p^m - p^0}{\Delta t} + D_x^x \otimes D_y I_y \sum_r \theta_r^m u^r + D_x^x I_x \otimes D_y \sum_r \theta_r^m v^r),$$

$$T_v^{2,m}(\underline{\underline{q}}) = M_x \otimes M_y \frac{v^m - v^0}{\Delta t} + M_x \otimes D_y \sum_r \theta_r^m p^r +$$
$$\alpha h (M_x \otimes D^y \frac{p^m - p^0}{\Delta t} + D_x \otimes D_y^y I_y \sum_r \theta_r^m u^r + D_x I_x \otimes D_y^y \sum_r \theta_r^m v^r),$$

$$T_p^{2,m}(\underline{\underline{q}}) = M_x \otimes M_y \frac{p^m - p^0}{\Delta t} + D_x \otimes D_y I_y \sum_r \theta_r^m u^r + D_x I_x \otimes D_y \sum_r \theta_r^m v^r +$$
$$\alpha h (D^x \otimes M_y \frac{u^m - u^0}{\Delta t} + M_x \otimes D^y \frac{v^m - v^0}{\Delta t} + (D_x^x \otimes M_y + M_x \otimes D_y^y) \sum_r \theta_r^m p^r).$$