How to Preserve Moving Equilibria for Hyperbolic Balance Laws with the Global Flux Quadrature

Davide Torlo, Wasilij Barsukow, Mirco Ciallella, Mario Ricchiuto, Moussa Ziggaf

Dipartimento di Matematica “Guido Castelnuovo”, Università di Roma La Sapienza, Italy
davidetorlo.it

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SAPIENZA
UNIVERSITÀ DI ROMA

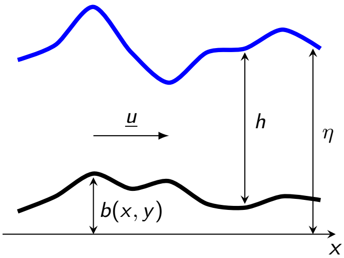
Water equilibria and perturbations

- Lake at rest perturbation
- Moving stationary wave
- Vortex type stationary solutions

Equilibria for shallow water equations

Shallow Water Equations

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y\left(huv\right) = -gh\partial_x b \\ \partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = -gh\partial_y b \end{cases}$$



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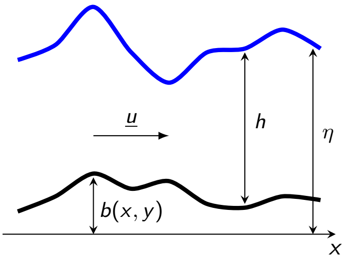
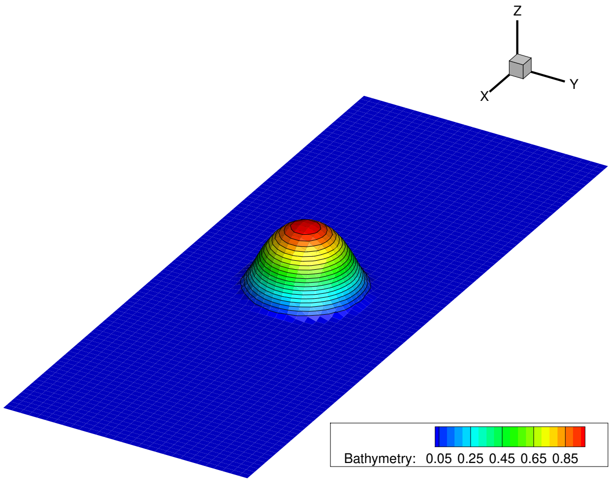


Lake at rest equilibrium

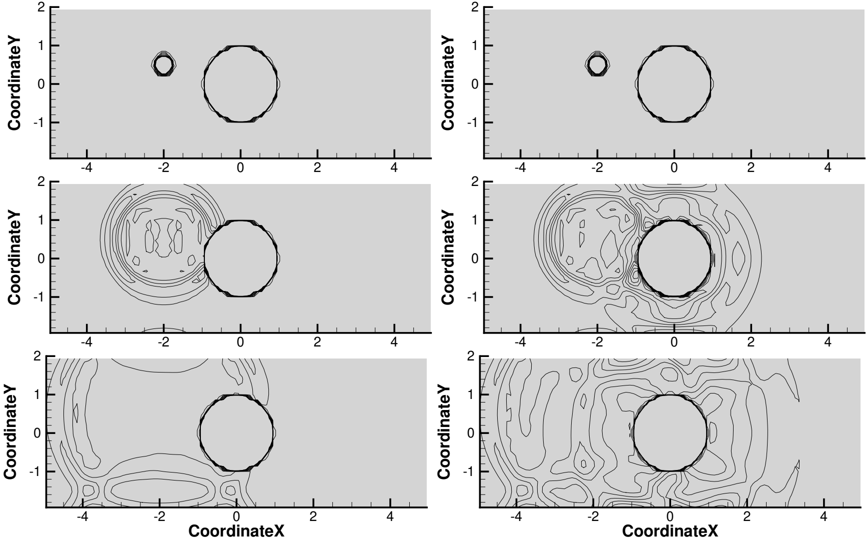
$$h(x,y) + b(x,y) \equiv \eta_0 \qquad u(x,y) = v(x,y) \equiv 0$$

$$\partial_x\left(\frac{g}{2}h^2\right) + gh\partial_x b = gh\partial_x h + gh\partial_x b = gh\partial_x \eta_0 = 0.$$

Simulation example lake at rest with perturbation



Simulation example lake at rest with perturbation



Equilibria for shallow water equations

Shallow Water Equations 1D

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{g}{2} h^2 \right) = -gh\partial_x b \end{cases}$$

Stationary waves in 1D

$$hu(x) =: q(x) \equiv q_0^x$$

and h such that

$$\partial_x \left(hu^2 + \frac{g}{2} h^2 \right) + gh\partial_x b = 0$$

$$\dots$$

$$\partial_x \left(\frac{q^2}{2gh^2} + h + b \right) = 0$$

$$\frac{q^2}{2gh^2(x)} + h(x) + b(x) = \mathcal{Q}(x_0) \tag{1}$$

Equilibria for shallow water equations

Shallow Water Equations 1D

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Cubic equation solutions

- Supercritical state $u > \sqrt{gh}$
- Subcritical state $u < \sqrt{gh}$
- Negative h

Stationary waves in 1D

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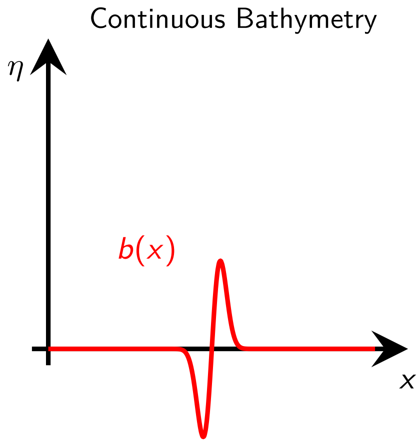
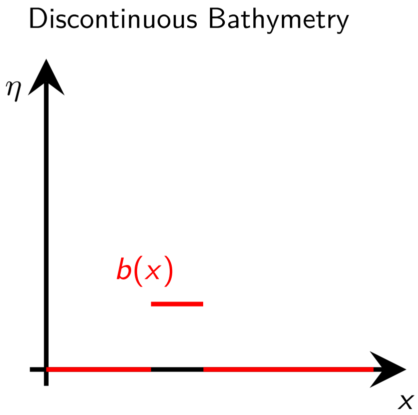
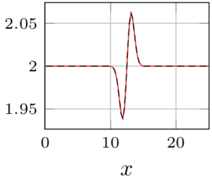
Simulation example moving equilibria non flat bathymetry



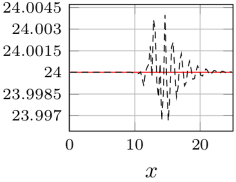


Simulation example moving equilibria non flat bathymetry

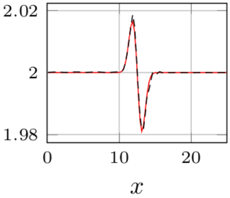
Continuous Bathymetry



(a) η



(b) q



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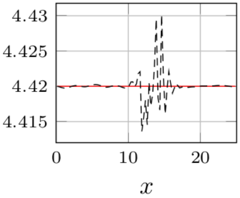
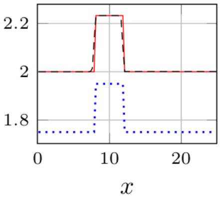


(b) q

Discontinuous Bathymetry



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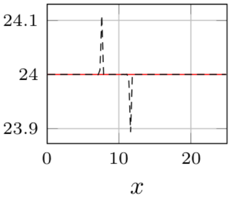
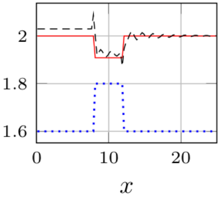


(b) q



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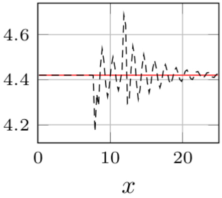


(b) q

2

Equilibria for shallow water equations

Shallow Water Equations (no bathymetry)

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) + \partial_y\left(huv\right) = 0 \\ \partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{g}{2}h^2\right) = 0 \end{cases}$$

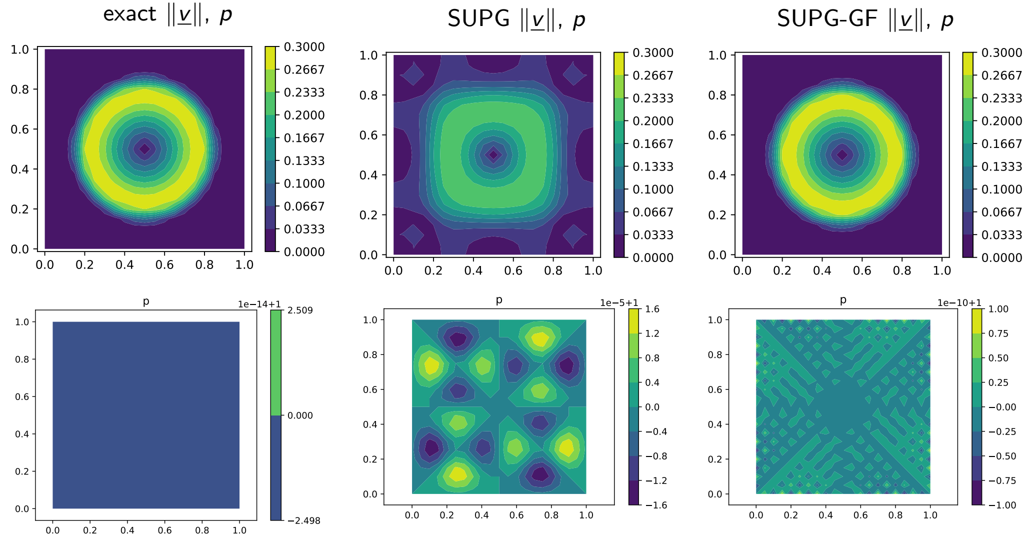
Vortices: Div-free solutions

$$\begin{cases} r = (x-x_0)^2 + (y-y_0)^2 \quad \theta = \arctan\left(\frac{y-y_0}{x-x_0}\right) \\ u(r) = -\sin(\theta)u_\theta(r) \qquad v(r) = \cos(\theta)u_\theta(r) \\ h(r): \quad h'(r)gr = u_\theta^2(r) \end{cases}$$

Other equations

- Euler equations (isentropic)
- Linear Acoustic equations

Simulation example of a vortex (for linear acoustics)



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How can we preserve the equilibria?

Exactly!

Impossible: discretization of data b, of the solutions h, u, v

Exactly with respect to discretization

- Possible
- Might involve some analytical equation to be solved
- Requires the knowledge a priori of equilibria type

Exactly Well Balancing

Better than the underlying method

- Possible
- No need of inverting analytical equations
- No need of a priori knowledge of the equilibrium type

Well Balancing

Global Flux

Global Flux

- Obtain 1 differential operator for everything
- Put together flux and source
- Integrate the forms
- Gascón 2001[a], Chertock 2022[b], Ciallella 2023[c], Barsukow 2024[d]

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[a] Gascón, L., Corberán, J. J. Comput. Phys. 172(1), 261–297 (2001)

[b] Chertock, A., Kurganov, A., Liu, X., Liu, Y., & Wu, T. (2022). Journal of Scientific Computing, 90, 1-21.

[c] Ciallella, M., Torlo, D., & Ricchiuto, M. (2023). Journal of Scientific Computing, 96(2), 53.

[d] Barsukow, W., Ricchiuto, M., & Torlo, D. (2025). Numerical Methods for Partial Differential Equations 41.1 (2025): e23167.

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1D source recipe

$$\partial_t V + \partial_x f(V) = S(V, x)$$
$$\partial_t V + \partial_x (f(V) - K(V, x)) = 0$$
$$K(V, x) := \int_{x_0}^{x} S(V(s), s) ds$$

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1D source recipe

$$\begin{aligned}
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&\partial_t V + \partial_x (f(V) - K(V,x)) = 0\\
&K(V,x) := \int_{x_0}^{x} S(V(s),s)ds
\end{aligned}$$

2D divergence recipe

$$\begin{aligned}
&\partial_t h + \partial_x f + \partial_y g = 0, \qquad f = hu, \quad g = hv,\\
&\partial_t h + \partial_{xy}(F+G) = 0\\
&F(x,y) := \int_{y_0}^{y} f(x,\xi)d\xi, \quad G(x,y) := \int_{x_0}^{x} g(\xi,y)d\xi.
\end{aligned}$$

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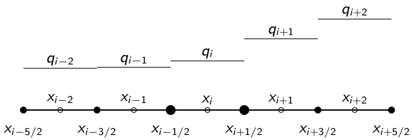
Global Flux in 1D for FV 1st order

$$\partial_t q + \partial_x f(q) = S(q,x) \qquad \Longrightarrow \qquad \partial_t q + \partial_x G(q,x) = 0$$

$$\text{with } G(q,x) := f(q) - K(q,x) = f(q) - \int_{x_0}^{x} S(q(s),s)\mathrm{d}s.$$

$$\text{FV: } q_i \approx \int_{x_{i-1/2}}^{x_{i+1/2}} q(x)\mathrm{d}x$$



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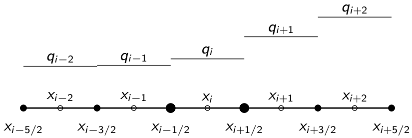

$$f_i := \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} f(q(x,t))\mathrm{d}x,$$

$$K_i \approx K(x_i, q(x_i)) = \int_{x_0}^{x_i} S(q(s),s)\mathrm{d}s \approx K_{i-1} + \int_{x_{i-1}}^{x_i} S(q(s),s)\mathrm{d}s,$$

$$K_i := K_{i-1} + \Delta x \frac{S_{i-1} + S_i}{2}$$

$$G_i := f_i - K_i.$$

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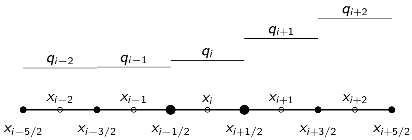

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$$G_i := f_i - K_i.$$

Numerical flux depends only on G: upwind, Roe, NO Rusanov

$$\partial_t q_i + \frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} = 0,$$

$$\hat{G}_{i+1/2} := \mathrm{sign}(J)^+ G_i + \mathrm{sign}(J)^- G_{i+1},$$

Global Flux in 1D for FV 1st order

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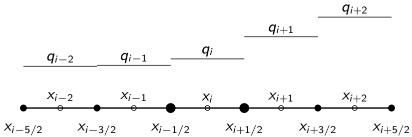

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$$\hat{G}_{i+1/2} := \mathrm{sign}(J)^+ G_i + \mathrm{sign}(J)^- G_{i+1},$$

Equilibrium: $\hat{G}_{i+1/2} = \hat{G}_{i-1/2} = \hat{G}_0$ for all i

$$f_i - K_i = G_0$$

Mind: high order, other equilibria (LAR), super convergence

Developing GF 1D FV 1st order

I want you to hate me, let's do the computations in a simple case (upwind)!

Formulae

- $\partial_t q_i = -\frac{\hat{G}_{i+1/2}-\hat{G}_{i-1/2}}{\Delta x}$
- $G_i = f_i - K_i$
- $K_i = K_{i-1} + \Delta x \frac{S_{i-1}+S_i}{2}$
- $\operatorname{sign}(J) = +1$
- $\hat{G}_{i+1/2} = G_i$

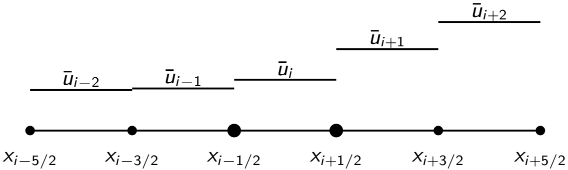
Classical Upwind FV

$$\partial_t q_i = -\frac{f_i - f_{i-1}}{\Delta x} + S_i$$

Expand!

$$\begin{aligned}
\partial_t q_i &= -\frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} \\
&= -\frac{G_i - G_{i-1}}{\Delta x} \\
&= -\frac{f_i - f_{i-1}}{\Delta x} + \frac{K_i - K_{i-1}}{\Delta x} \\
&= -\frac{f_i - f_{i-1}}{\Delta x} + \frac{S_{i-1} + S_i}{2}.
\end{aligned}$$

High order WENO GF [1]



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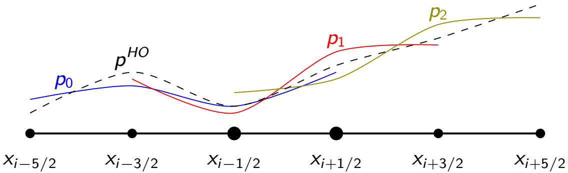

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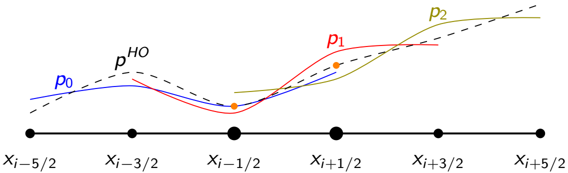

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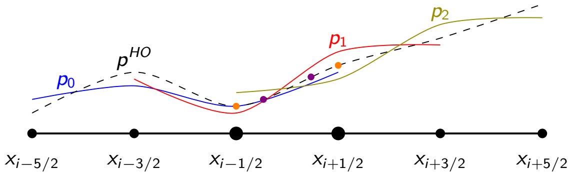

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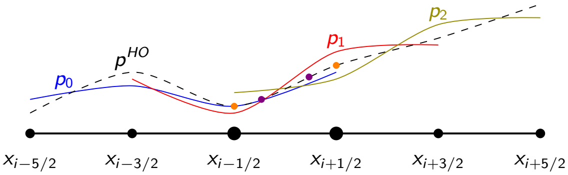

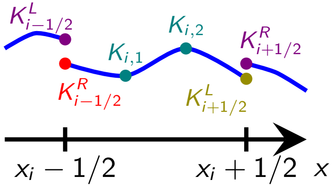

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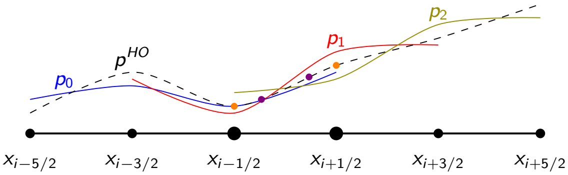

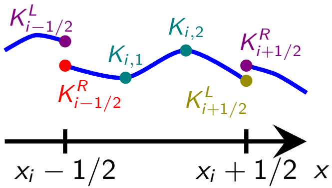

Global Flux Reconstruction

- Compute recursively K in quadrature points and interfaces (maybe also jump of K)
- Reconstruct in all quadrature points
  - Flux $f_{i,\theta}$
  - Integral of the source $K_{i,\theta}$
  - Global fluxes $G_{i,\theta} := f_{i,\theta} + K_{i,\theta}$
- Compute the cell average of the global flux G
- Well balancing for lake at rest

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Validation: Subcritical flow and perturbation

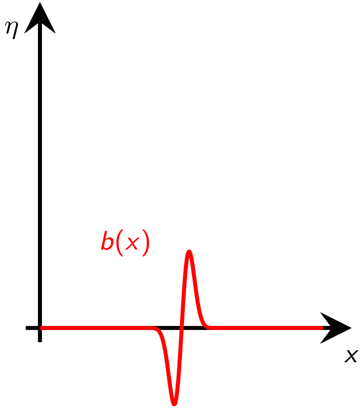
Domain and Bathymetry

$$\begin{aligned}\Omega &= [0, 25],\\ b(x) &= 0.05 \sin(x - 12.5) \exp\left(1 - (x - 12.5)^2\right),\\ g &= 9.812.\end{aligned}$$

$b(x)$ is chosen $\mathcal{C}^\infty$ and such that it has values smaller than machine precision at the boundaries.

Subcritical flow test

$$\begin{aligned}&\text{IC: } h(x,0) = 2 - b(x), && q(x,0) \equiv 0,\\ &\text{BC: } h(25,t) = 2, && q(0,t) = 4.42,\end{aligned}$$



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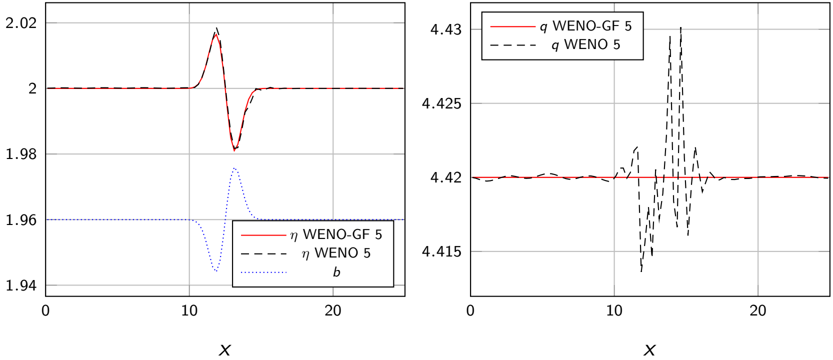
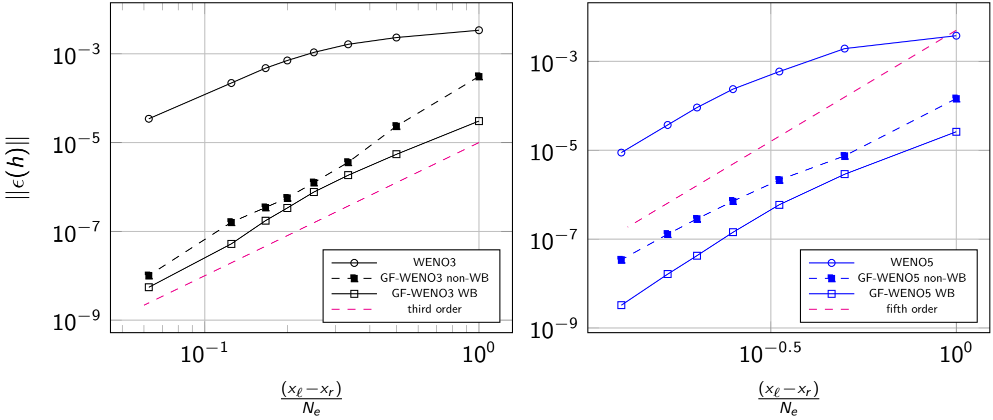


(a) η (b) q

Figure: Subcritical flow: characteristic variables compute by means of the GF-WENO5 (red continuous line) and WENO5 (black dashed line) schemes with $N_e = 100$.

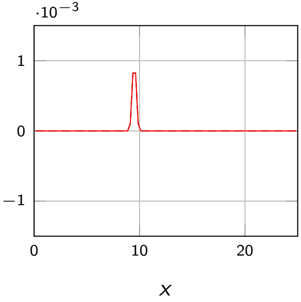
Validation: Subcritical flow and perturbation



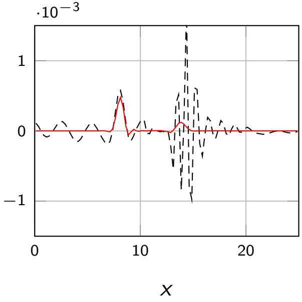
(a) WENO3 (b) WENO5

Figure: Subcritical flow: convergence tests with WENO3 and WENO5.

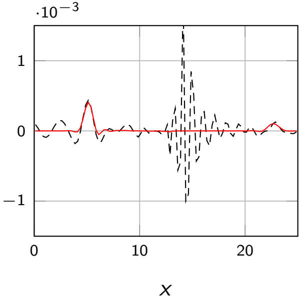
Validation: Subcritical flow and perturbation



(a) $t = 0$



(b) $t = 0.66$



(c) $t = 2$

Figure: Perturbation on a subcritical flow: $\eta - \eta^{eq}$

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FEM+SUPG GF 2D high order

Acoustics

$$\partial_t u + \partial_x p = 0$$
$$\partial_t v + \partial_y p = 0$$
$$\partial_t p + \partial_x u + \partial_y v = 0$$

FEM+SUPG GF 2D high order

SUPG FEM for acoustics

$$\int \varphi(\partial_t u + \partial_x p) + \alpha\Delta\, \partial_x\varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha\Delta\, \partial_y\varphi(\partial_t p + \partial_x u + \partial_y v) = 0$$

$$\int \varphi(\partial_t p + \partial_x u + \partial_y v) + \alpha\Delta\, \partial_x\varphi(\partial_t u + \partial_x p) + \alpha\Delta\, \partial_y\varphi(\partial_t v + \partial_y p) = 0$$

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Acoustics

$$\partial_t u + \partial_x p = 0$$
$$\partial_t v + \partial_y p = 0$$
$$\partial_t p + \partial_x u + \partial_y v = 0$$

Details on discretization

- Cartesian grid!!
- Gauss-Lobatto points for quadrature and Lagrange basis function
- Explicit arbitrary high order time discretization with Deferred Correction

Global flux in 2D

Main idea

Define

$$\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds \qquad \boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$$

So that

$$\partial_t p + \partial_x u + \partial_y v = \partial_t p + \partial_{xy}(\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0$$

Global flux in 2D

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Global Flux SUPG for acoustics

Define $\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds$ and $\boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$, with $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y \in V_h^K(\Omega_h)$, $\Phi := \boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y$.

Global flux in 2D

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Global Flux SUPG for acoustics

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$$\int \varphi(\partial_t u + \partial_x p) + \alpha \Delta x\, \partial_x \varphi(\partial_t p + \partial_x \partial_y \Phi) = 0$$

$$\int \varphi(\partial_t v + \partial_y p) + \alpha \Delta y\, \partial_y \varphi(\partial_t p + \partial_x \partial_y \Phi) = 0$$

$$\int \varphi(\partial_t p + \partial_x \partial_y \Phi) + \alpha \Delta x\, \partial_x \varphi(\partial_t u + \partial_x p) + \alpha \Delta y\, \partial_y \varphi(\partial_t v + \partial_y p) = 0$$

Global Flux SUPG: changes

Global Flux SUPG for acoustics

$\boldsymbol{\sigma}_x(x,y) := \int_{y_0}^{y} u(x,s)ds$ and $\boldsymbol{\sigma}_y(x,y) := \int_{x_0}^{x} v(s,y)ds$, with $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y \in V_h^K(\Omega_h)$.

Changes in equilibrium

$$\begin{aligned} &\nabla \cdot \underline{v} = 0\\ \Longrightarrow &\partial_x\partial_y(\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y) = 0\\ \Longleftrightarrow &\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y = f(x) + g(y) \end{aligned}$$

Discrete equilibrium

$$\begin{aligned} &\partial_x\partial_y\Phi(x_i,y_j) = 0\\ \Longrightarrow &\int_{x_0}^{x_i}\int_{y_0}^{y_j} \partial_y\partial_x\Phi(x,y)dxdy = 0 \quad \forall i,j\\ \Longrightarrow &\int_{x_0}^{x_i} \partial_x\Phi(x,y_j)dx - \int_{x_0}^{x_i} \partial_x\Phi(x,y_0)dx = 0 \quad \forall i,j\\ \Longrightarrow &\Phi(x_i,y_j) - \Phi(x_0,y_j) - \Phi(x_i,y_0) + \Phi(x_0,y_0) = 0 \quad \forall i,j\\ \Longrightarrow &\Phi(x_i,y_j) = f_i + g_j \end{aligned}$$

Global Flux is not global!

- In principle $\boldsymbol{\sigma}_x(x,y) = \int_{y_B}^{y} u(x,s)ds$ should be integrated from the beginning (bottom) of the domain y_B!
- In practice we always use $\partial_x \partial_y \boldsymbol{\sigma}_x(x,y)$ integrated in one cell!!!!
- So,

$$\boldsymbol{\sigma}_x(x,y) = \int_{y_B}^{y} u(x,s)ds = \underbrace{\int_{y_B}^{y_0} u(x,s)ds}_{\text{constant in one cell!}} + \int_{y_0}^{y} u(x,s)ds$$

  whatever constant we bring from outside the cell, is canceled out

$$\partial_y \boldsymbol{\sigma}_x(x,y) = \partial_y \int_{y_B}^{y} u(x,s)ds = \partial_y \int_{y_B}^{y_0} u(x,s)ds + \partial_y \int_{y_0}^{y} u(x,s)ds = \partial_y \int_{y_0}^{y} u(x,s)ds$$

- At the discrete level we have an integral operator I_y and a differential operator D_y that together give a weird averaging operator $D_y I_y$

Coriolis and sources

Extension to source terms

$$\partial_t u + \partial_x p = S_u$$
$$\partial_t v + \partial_y p = S_v$$
$$\partial_t p + \partial_x u + \partial_y v = S_p$$

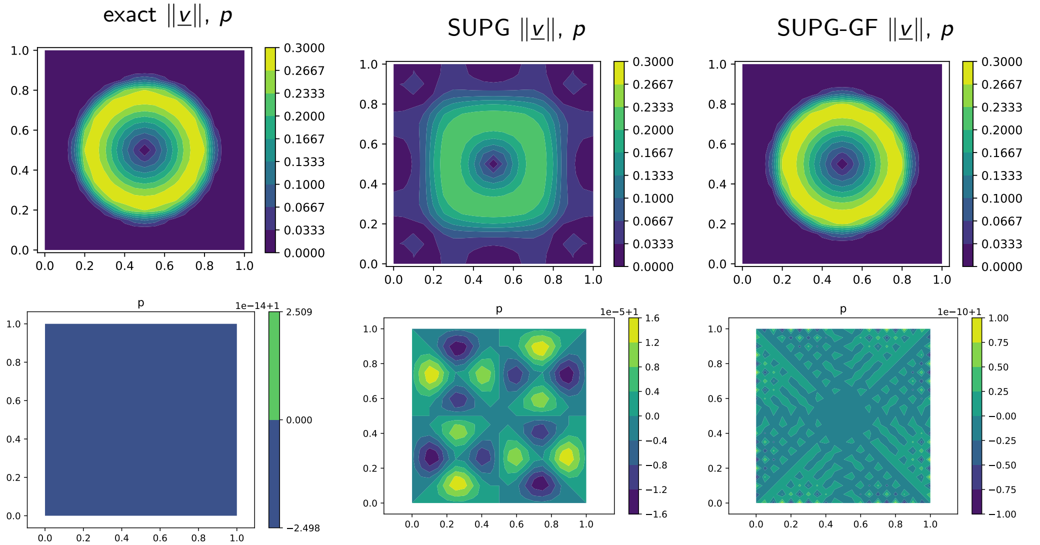
Source terms

- Coriolis
- Mass sources
- Friction

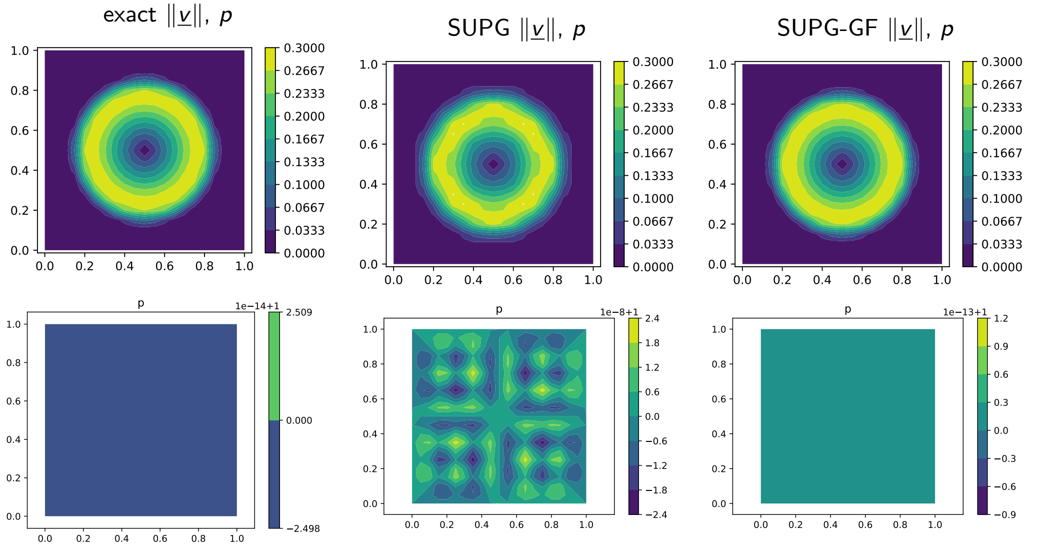
Global flux for sources

$$\mathcal{G}_u := p - \int^x S_u$$
$$\mathcal{G}_v := p - \int^y S_v$$
$$\mathcal{G}_p := \int^y u + \int^x v - \int^x \int^y S_u$$

Simulation of vortex (linear acoustics) FEM+SUPG: $\mathbb{Q}^1$, $N_x = N_y = 20$



Simulation of vortex (linear acoustics) FEM+SUPG: $\mathbb{Q}^2$, $N_x = N_y = 10$



Simulation of vortex: errors

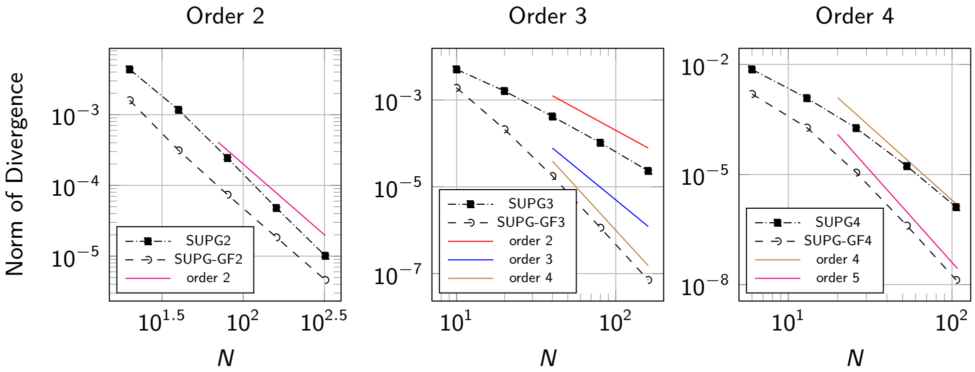


Figure: Smooth vortex: convergence of L^2 error of u with respect to the number of elements in x

Vortex perturbation

Pressure perturbation

- Gaussian centered in $\underline{x}_p = (0.4, 0.43)$
- scaling coefficient $r_0 = 0.1$
- radius $\rho(\underline{x}) = \sqrt{||\underline{x} - \underline{x}_p||}/r_0$

$$\delta_p(\underline{x}) = \varepsilon e^{-\frac{1}{2(1-\rho(\underline{x}))^2}+\frac{1}{2}},$$

- final time $T = 0.35$

Vortex perturbation

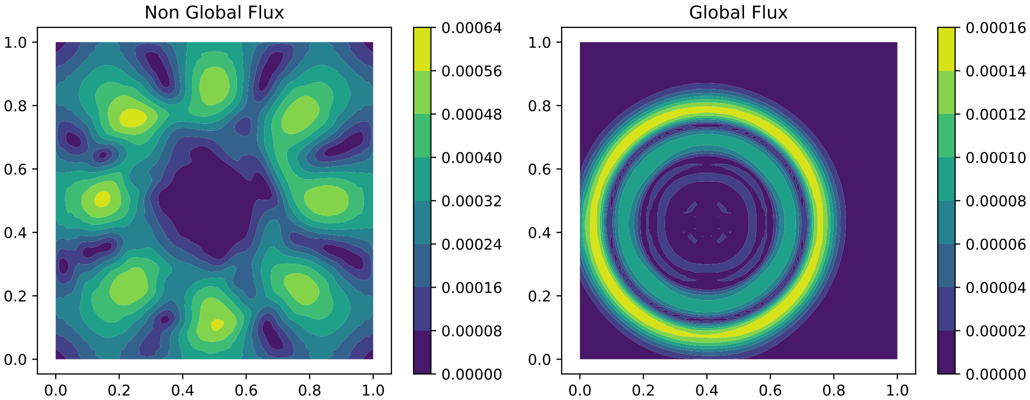


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^1$ with 80×80 cells and 6561 dofs.

Vortex perturbation

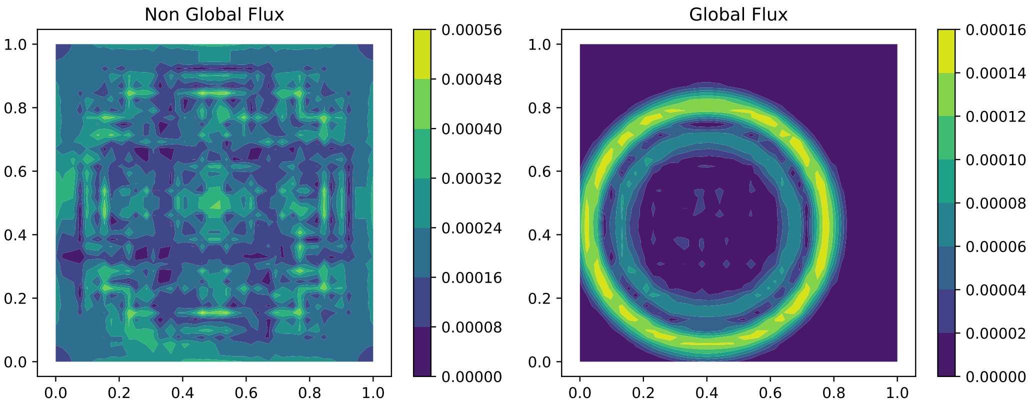


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 13×13 cells and 1600 dofs.

Vortex perturbation

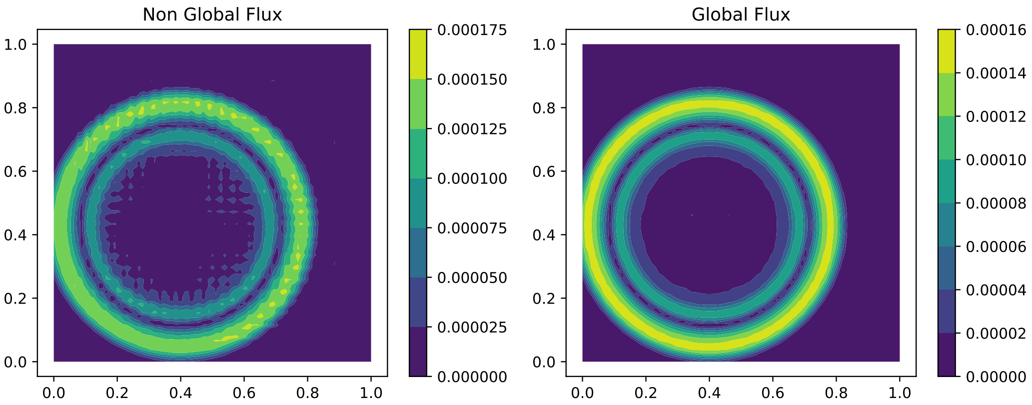


Figure: Perturbation($\varepsilon = 10^{-3}$) test. Plot of $\|\underline{u}_{eq} - \underline{u}_p\|$, with $\underline{u}_{eq}$ the equilibrium obtained with a cheap optimization process. $\mathbb{P}^3$ with 26 cells and 6241 dofs.

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FEM+SUPG for GF

Discretizations

- GF+SUPG+FEM works easily also for nonlinear problems on paper
- GF+FV less trivial, because ...

GF+FEM+SUPG

$$\partial_t u + \partial_x F(u) + \partial_y G(u) = S(u) \Longrightarrow \partial_t u + \partial_x \partial_y \mathcal{G}(u) = 0$$

$$\mathcal{G}(u) := \int^y F(u) + \int^x G(u) - \int^x \int^y S(u);$$

$$\int_\Omega (\varphi + \alpha \Delta \partial_x \varphi J^x + \alpha \Delta \partial_y \varphi J^y)(\partial_t u + \partial_{xy} \mathcal{G}(u)) = 0 \qquad \forall \varphi.$$

GF+FEM+FV

As FEM+SUPG, but on the dual mesh (corner flux)

Euler equations: isentropic vortex (steady state)

IC

$$(\rho, u, v, p) = (1 + \delta\rho,\, \delta u,\, \delta v,\, 1 + \delta p)\,.$$

The test case is set up in a $[0,10] \times [0,10]$ domain with periodic boundary conditions and vortex radius $r = \sqrt{(x-5)^2 + (y-5)^2}$. The vortex strength is $\epsilon = 5$, and the entropy perturbation is assumed to be zero. Given these hypothesis, the perturbations on velocity and temperature can be written as

$$\begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = \frac{\epsilon}{2\pi} \exp\left(\frac{1-r^2}{2}\right) \begin{bmatrix} -(y-5) \\ (x-5) \end{bmatrix}, \qquad \delta T = -\frac{(\gamma-1)\epsilon^2}{8\gamma\pi^2} \exp(1-r^2).$$

It follows that the perturbations on density and pressure reads

$$\delta\rho = (1+\delta T)^{\frac{1}{\gamma-1}} - 1, \qquad \delta p = (1+\delta T)^{\frac{\gamma}{\gamma-1}} - 1.$$

Euler equations: isentropic vortex (steady state)

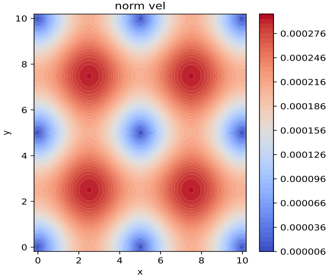
Euler equations: isentropic vortex ($t_f = 1$). L_2 error and order of accuracy $\tilde{n}$ for FV-1, FV-2 and GF methods.

N_x, N_y	ρ		ρu		ρv		ρE	
	L_2	$\tilde{n}$	L_2	$\tilde{n}$	L_2	$\tilde{n}$	L_2	$\tilde{n}$
FV-1								
20	3.58E-01	–	6.77E-01	–	6.77E-01	–	1.16E+00	–
40	2.47E-01	0.53	4.40E-01	0.62	4.40E-01	0.62	8.29E-01	0.48
80	1.49E-01	0.72	2.59E-01	0.76	2.59E-01	0.76	5.15E-01	0.68
160	8.33E-02	0.84	1.43E-01	0.85	1.43E-01	0.85	2.91E-01	0.82
320	4.42E-02	0.91	7.56E-02	0.91	7.56E-02	0.91	1.56E-01	0.90
FV-2								
20	1.06E-01	–	2.05E-01	–	2.00E-01	–	4.32E-01	–
40	3.62E-02	1.55	6.74E-02	1.60	6.71E-02	1.57	1.20E-01	1.85
80	1.07E-02	1.76	1.93E-02	1.80	1.95E-02	1.78	2.91E-02	2.04
160	2.39E-03	2.16	5.58E-03	1.78	5.61E-03	1.79	7.04E-03	2.04
320	5.12E-04	2.22	1.39E-03	2.00	1.39E-03	2.01	1.56E-03	2.17
GF								
20	1.52E-02	–	3.67E-02	–	3.67E-02	–	4.59E-02	–
40	5.95E-03	1.35	1.15E-02	1.67	1.15E-02	1.67	1.54E-02	1.57
80	1.76E-03	1.76	3.06E-03	1.90	3.06E-03	1.90	4.35E-03	1.82
160	4.69E-04	1.90	7.87E-04	1.96	7.87E-04	1.96	1.16E-03	1.90
320	1.21E-04	1.95	2.00E-04	1.97	2.00E-04	1.97	3.02E-04	1.94

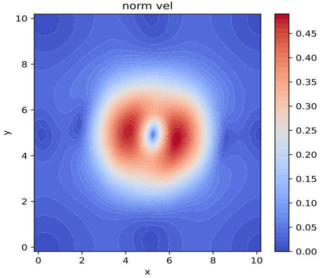
Euler equations: isentropic vortex (steady state)

Euler equations: isentropic vortex. Isocontours of the velocity norm obtained with FV-1, FV-2 and GF after a long time integration ($t_f = 200$)

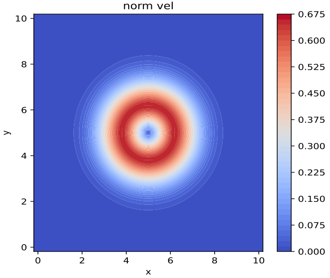
FV-1



FV-2

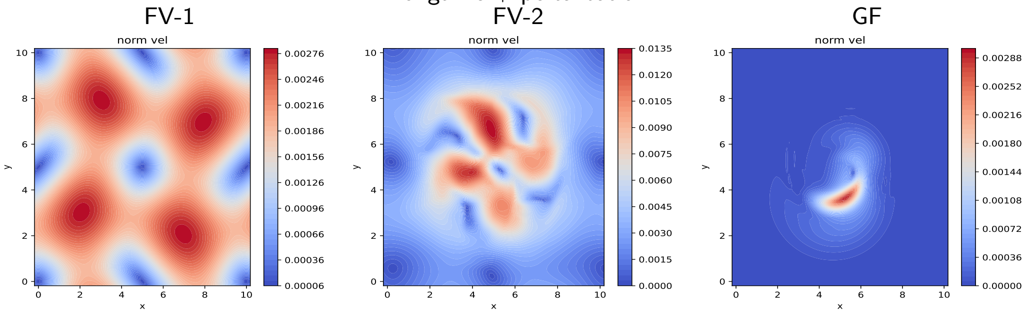


GF



Euler equations: isentropic vortex (steady state)

Euler equations: perturbation of the isentropic vortex. Isocontours of the $\rho - \rho_{\text{eq}}$ norm obtained with FV-1, FV-2 and GF at final time $t_f = 2$ with a 80×80 mesh. Take as IC the final simulation of longtime + perturbation



Euler equations: Kevin-Helmoltz instability

- Domain $[0,2] \times [-1/2, 1/2]$
- Final time $t_f = 80$
- initial condition

$$\rho = \gamma + \mathcal{H}(y)\, r, \qquad u = M\,\mathcal{H}(y), \qquad v = \delta\, M \sin(2\pi x), \qquad p = 1,$$

- Mach number parameter $M = 10^{-2}$, $r = 10^{-3}$, $\delta = 0.1$
- $\mathcal{H}(y)$

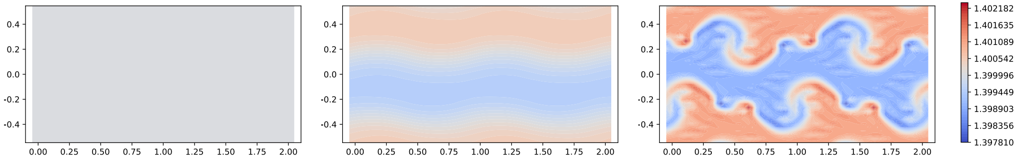
$$\mathcal{H}(y) = \begin{cases} -\sin\left(\frac{\pi}{\omega}\left(y+\frac{1}{4}\right)\right), & \text{if} \quad -\frac{1}{4}-\frac{\omega}{2} \le y < -\frac{1}{4}+\frac{\omega}{2}, \\ -1, & \text{if} \quad -\frac{1}{4}+\frac{\omega}{2} \le y < \frac{1}{4}-\frac{\omega}{2}, \\ \sin\left(\frac{\pi}{\omega}\left(y-\frac{1}{4}\right)\right), & \text{if} \quad \frac{1}{4}-\frac{\omega}{2} \le y < \frac{1}{4}+\frac{\omega}{2}, \\ 1 & \text{else}, \end{cases}$$

where $\omega = 1/16$.

Euler equations: Kevin-Helmoltz instability

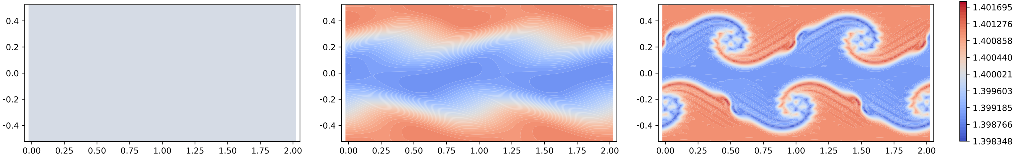
$N_x = 64, \ N_y = 32$

FV-1 FV-2 GF



$N_x = 128, \ N_y = 64$

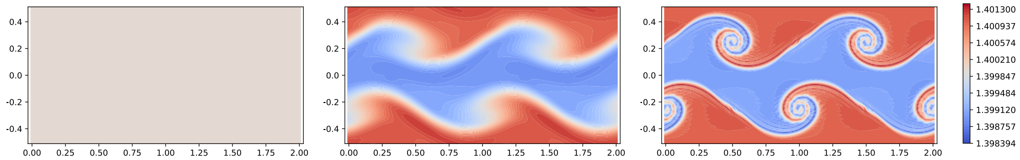
FV-1 FV-2 GF



Euler equations: Kevin-Helmoltz instability

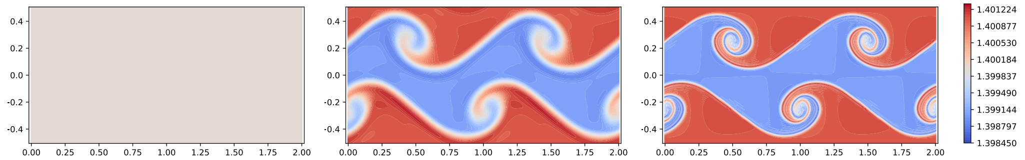
$$N_x = 256,\ N_y = 128$$

FV-1 FV-2 GF

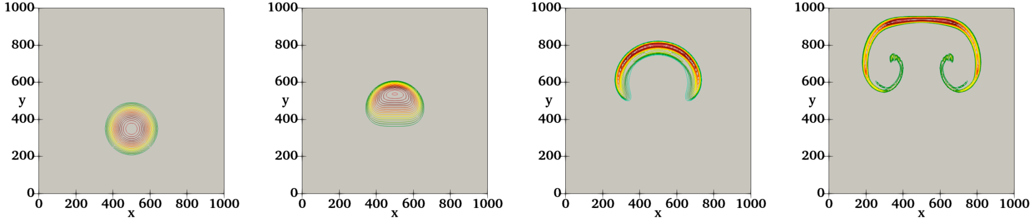


$$N_x = 512,\ N_y = 256$$

FV-1 FV-2 GF



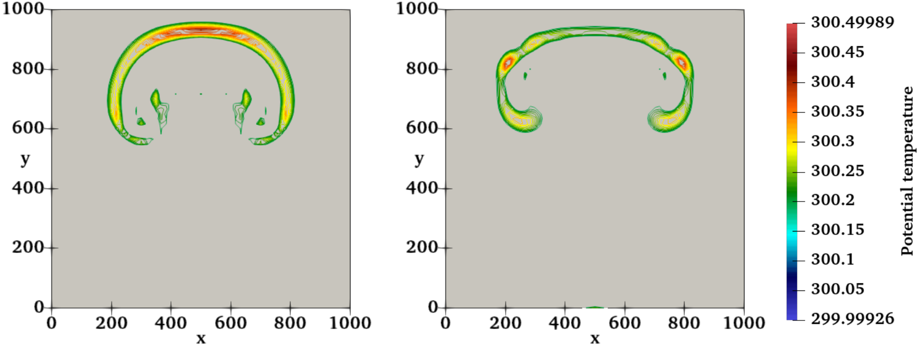
Euler equations, FEM: Thermal rising bubble



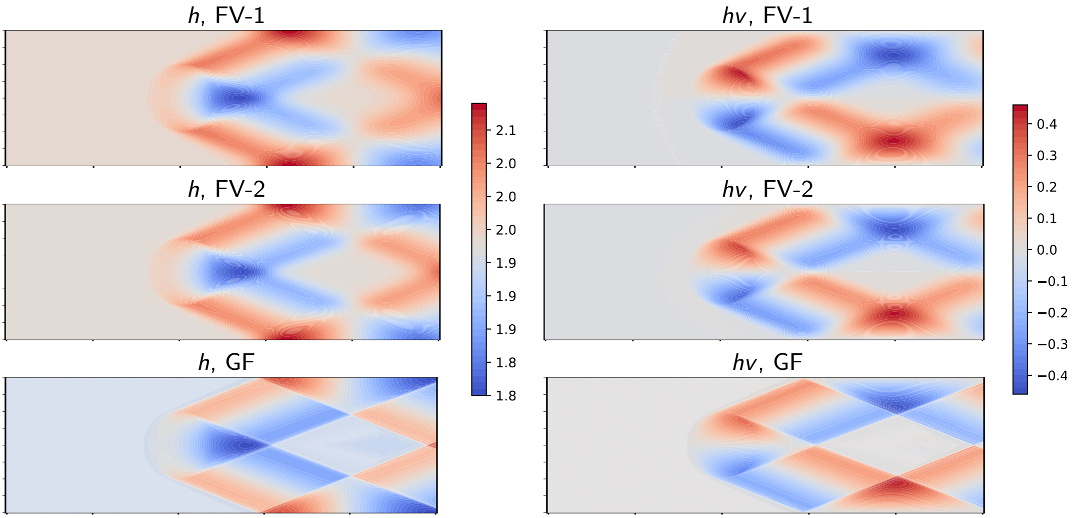
Top: GF 150x150

Bottom left: GF 60x60

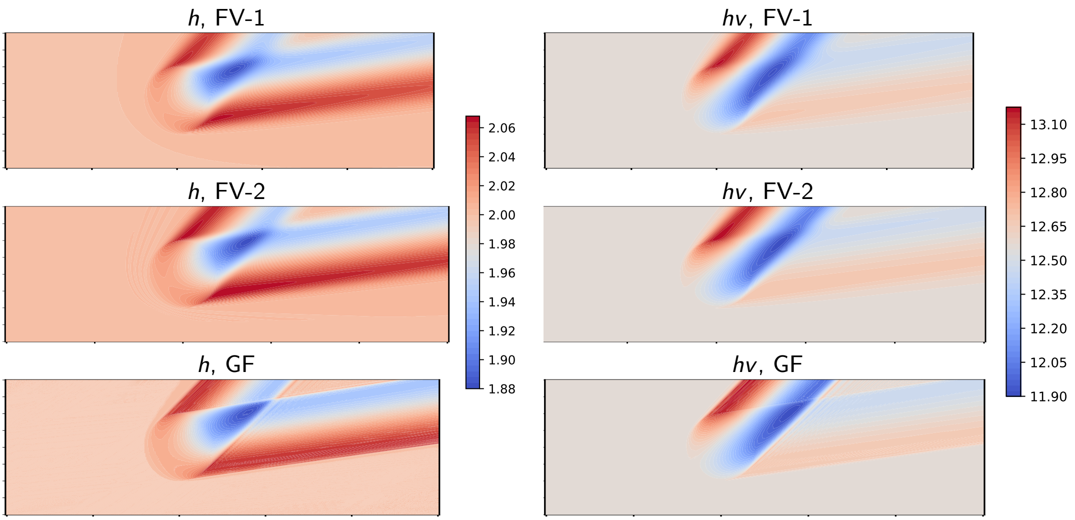
Bottom right: SUPG 60x60



Shallow water: subcritical flow with bathymetry

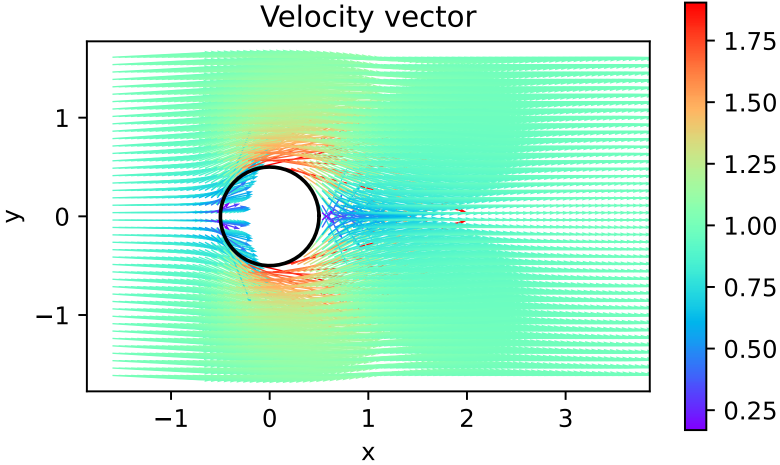


Shallow water: subcritical flow with bathymetry



Acoustics equations: potential flow around a cylinder

Using immersed boundary method (IBM) based on extrapolation



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Extensions and Perspectives

Summary

- Global Flux to preserve moving equilibria
- 1D integrate the source and unique flux
- 2D integrate F in y and G in x
- Some superconvergence in steady states
- Extra accuracy in vorticity like problems
- Small stability issues with very very long time simulations in nonlinear 2D
- No problem with shocks (we were surprised)

Perspectives

- Very long time behavior weak instability for vortices
- Other methods: DG seems less trivial
- Immersed Boundary Method developments
- Riemann solver for corner problems?
- Non Cartesian meshes

Extensions and Perspectives

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THANKS!!

State of the art techniques (part 1)

Subtract equilibrium

- Know analytical equilibrium
- Dedner 2004[a] and Berberich 2021[b]

[a] Dedner, A., Rohde, C., Schupp, B., & Wesenberg, M. (2004). Computing and Visualization in Science, 7(2), 79-96.

[b] J. P. Berberich, P. Chandrashekar, and C. Klingenberg. Computers & Fluids, 219:104858, 2021.

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq} := (h^{eq}, u^{eq}, v^{eq})$
- Discrete exact equilibrium residual: $\mathcal{S}^{eq}(t^n) := \mathcal{S}(V^{eq}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq}(t^n)$

State of the art techniques (part 1)

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- Equilibrium: $V^{eq} := (h^{eq}, u^{eq}, v^{eq})$
- Discrete exact equilibrium residual: $\mathcal{S}^{eq}(t^n) := \mathcal{S}(V^{eq}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq}(t^n)$

Properties

☺ Ridiculously well balanced: $V^n = V^{eq} \Longrightarrow V^{n+1} = V^{eq}$

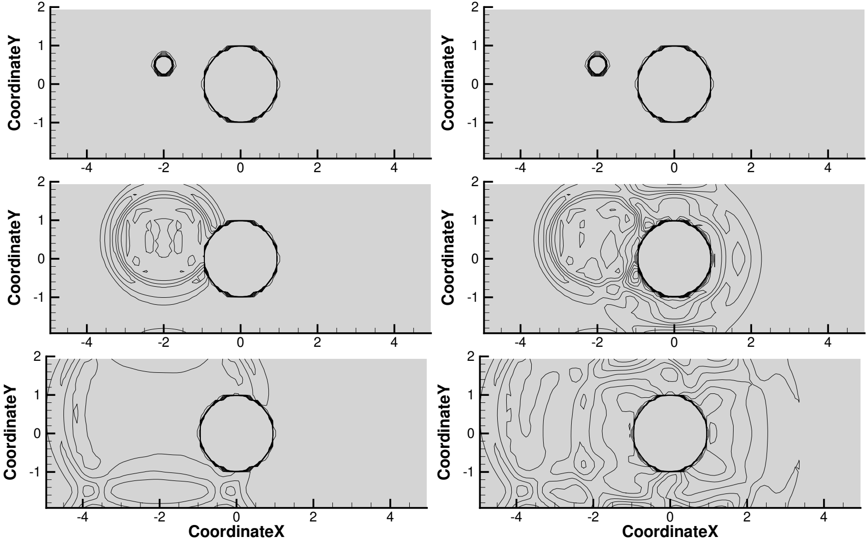
☹ Know equlibrium a priori

☺ Lake at rest

☹ Stationary waves

☹ 2D vortices

Example: subtract equilibrium[2]



[2]Ciallella, M., Micalizzi, L., Öffner, P., & Torlo, D. (2022). Computers & Fluids, 247, 105630.

State of the art techniques (part 2)[3]

Equilibrium reconstruction

- In every cell solve an ODE at reconstruction/quadrature points, constrained with the state V^n (BVP)
- ODE solver either exact or very accurate
- Malaga school

Procedure

- Base Scheme: $V^{n+1} = V^n + \mathcal{S}(V^n)$
- Equilibrium: $V^{eq,ODE}$:=ODE_Solver(1) subject to V^n
- Discrete equilibrium residual: $\mathcal{S}^{eq,ODE}(t^n) := \mathcal{S}(V^{eq,ODE}(t^n))$
- Well balanced scheme : $V^{n+1} = V^n + \mathcal{S}(V^n) - \mathcal{S}^{eq,ODE}(t^n)$

[3] Castro, M. J., & Parés, C. (2020). Journal of Scientific Computing, 82(2), 48.

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Properties

☺ Exactly well-balanced $V^n = V^{eq,ODE} \Longrightarrow V^{n+1} = V^{eq,ODE}$

☺ For all equilibria of one type

☹ Expensive (ODE solver for each cell)

☺ Lake at rest

☺ Stationary waves

☹ Problem for transcritical flows $u = \sqrt{gh}$

☹ 2D vortices

[3] Castro, M. J., & Parés, C. (2020). Journal of Scientific Computing, 82(2), 48.

State of the art techniques (part 3)[4]

Riemann problem modification

- For FV schemes
- Change the Riemann problem approximation
- Exploit (1) such that at equilibrium it is satisfied by the Riemann problem
- Michel-Dansac 2016

Properties

- Exactly well-balanced (if (1) analytically invertible else accurate solver) $V^n = V^{eq,ODE} \implies V^{n+1} = V^{eq,ODE}$
- ☺ For all equilibria of one type
- ☹ Computations by hand for Riemann Solver
- ☹ Only 1st order, blending with high order
- ☺ Lake at rest
- ☺ Stationary waves
- ☹ Problem for transcritical flows $u = \sqrt{gh}$
- ☹ 2D vortices

[4] Michel-Dansac, V., Berthon, C., Clain, S., & Foucher, F. (2016). Computers & Mathematics with Applications, 72(3), 568-593.

FV for GF 2D

GF+FV

$$\partial_t u + \partial_x \partial_y \mathcal{G}(u) = 0$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (\partial_t u + \partial_{xy} \mathcal{G}(u)) \mathrm{d}x \mathrm{d}y = 0$$

$$\partial_t u_{ij} + \hat{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} - \hat{\mathcal{G}}_{i-\frac{1}{2},j+\frac{1}{2}} - \hat{\mathcal{G}}_{i+\frac{1}{2},j-\frac{1}{2}} + \hat{\mathcal{G}}_{i-\frac{1}{2},j-\frac{1}{2}} = 0$$

Corner numerical flux!!

- Upwind didn't work for nonlinear 2D problems (it worked in 1D, it works for 2D linear acoustics, but for nonlinear 2D all tentative methods were unstable)
- We ended up with the same SUPG scheme, applied on the dual mesh of the FV

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Corner numerical flux: SUPG

$$\hat{\mathcal{G}}^{i+\ell,j+m}_{i+\frac{1}{2},j+\frac{1}{2}} = \overline{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} n^{(i+\ell,j+m)}_{i+\frac{1}{2},j+\frac{1}{2}} + \mathcal{D}^{(i+\ell,j+m)}_{i+\frac{1}{2},j+\frac{1}{2}},$$

$$\mathcal{D}^{(i+\ell,j+r)}_{i+\frac{1}{2},j+\frac{1}{2}} := \mathcal{D}(\widetilde{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}}, \bar{q}_{i+\frac{1}{2},j+\frac{1}{2}} | \mathbf{n}^{(i+\ell,j+r)}_{i+\frac{1}{2},j+\frac{1}{2}})$$

$$= \alpha\Delta \int_{\widetilde{C}} \left(\frac{1}{\Delta x} J^x \partial_\xi \phi_{\ell,r} + \frac{1}{\Delta y} J^y \partial_\eta \phi_{\ell,r} \right) \partial_{\xi\eta} \widetilde{\mathcal{G}}_{i+\frac{1}{2},j+\frac{1}{2}} \mathrm{d}\xi \mathrm{d}\eta,$$

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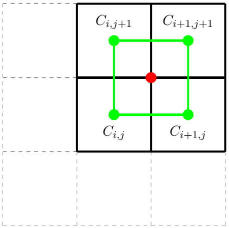
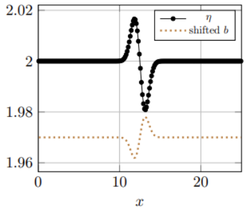
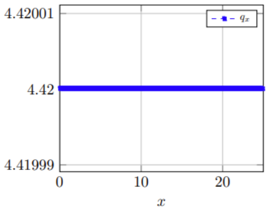


Example: Riemann Problem Change[5]

SUBCRITICAL

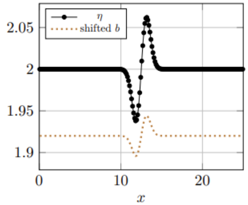


(a) free surface η and bathymetry b, shifted and rescaled



(b) discharge q_x

SUPERCRITICAL



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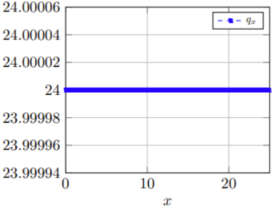

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[5] Ciallella, M., Micalizzi, L., Michel-Dansac, V., Öffner, P., & Torlo, D. (2025)

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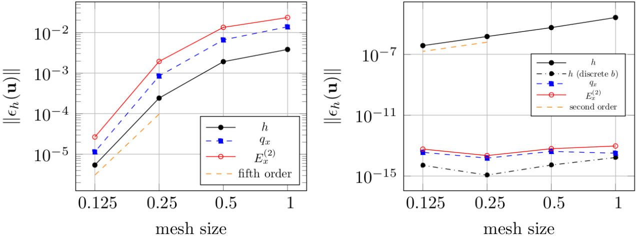
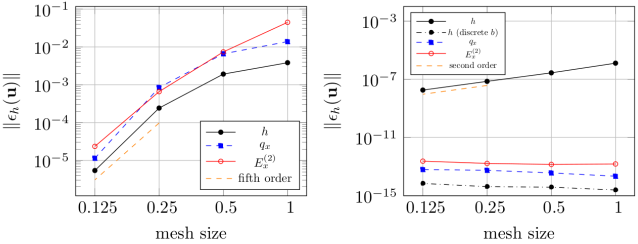


SUPERCRITICAL



[5] Ciallella, M., Micalizzi, L., Michel-Dansac, V., Öffner, P., & Torlo, D. (2025)

Vortex simulation: divergence error

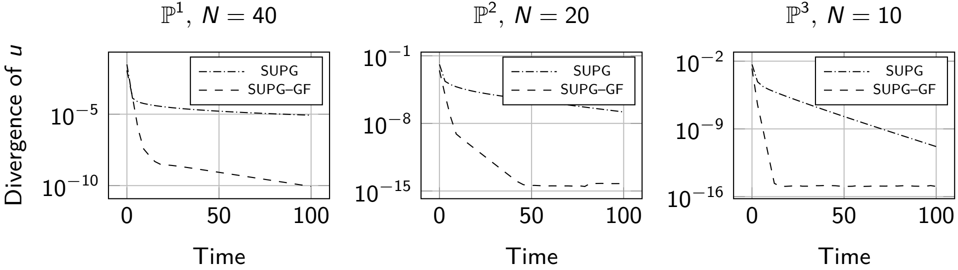


Figure: Norm of discrete divergence of $\underline{u}$ for SUPG $(\partial_x u + \partial_y v)$ and SUPG–GF $(\partial_x \partial_y (\boldsymbol{\sigma}_x + \boldsymbol{\sigma}_y))$ simulations with respect to time for different orders

Euler equations, FV: Low-Mach Shu Vortex